4. More about rational functions

Definition 4.1. *A* ***divisor*** D *on* $\mathbb{P}^1$ *is a formal linear combination of points with integral coefficients*

$$D = \sum_p n_p p,$$

where p *ranges over the points of* $\mathbb{P}^1$ *and only finitely many* $n_p \neq 0$.

The ***degree*** *of* D *is equal to* $\sum_p n_p \in \mathbb{Z}$.

Let $f(z) = P(z)/Q(z)$ *be a rational function. Suppose that we write* $f(1/z) = P_1(z)/Q_1(z)$. *The* ***divisor of zeroes of*** f *is*

$$(f)_0 = \sum_p \operatorname{ord}_p P(z)p + \operatorname{ord}_0 P_1(z)q$$

where $\operatorname{ord}_p P(z)$ *is the order of vanishing and* q *represents the point* $\infty = [1 : 0]$. *The* ***divisor of poles of*** f, *denoted* $(f)_\infty$, *is the divisor of zeroes of* $1/f$.

The ***divisor of*** f *is the formal difference*

$$(f) = (f)_0 - (f)_\infty.$$

The ***order*** *of* f *is equal to the degree of* $(f)_0$.

Proposition 4.2. *Let* $f(z) = P(z)/Q(z)$ *be a rational function. Then*

(1) *The degree of* (f) *is zero. That is, the number of zeroes equals the number of poles, counted according to multiplicity.*
(2) *The order of* f *is equal to the maximum of the degrees of* P *and* Q.
(3) *For every* $a \in \mathbb{C}$, *the order of* $f(z) - a$ *is equal to the order of* f. *That is, the number of solutions of the equation*

$$P(z)/Q(z) = a,$$

counted according to multiplicity, is equal to the order of f.

Proof. Suppose that the degree of P is m and the degree of Q is n. Then $z^m P(1/z)$ and $z^n Q(1/z)$ are both polynomials which are non-vanishing at zero. Assume that $m \leq n$. Now

$$f(1/z) = \frac{P(1/z)}{Q(1/z)} = \frac{z^n P(1/z)}{z^n Q(1/z)} = z^{n-m} \frac{z^m P(1/z)}{z^n Q(1/z)}.$$

Thus f has a zero of order $n - m$ at ∞. The case $m > n$ is similar. (1) and (2) follow.

We prove (3). The rational function

$$g(z) = f(z) - a = \frac{P(z)}{Q(z)} - a$$

has the same poles as f. Thus the degree of $(g)_\infty$ is the degree of $(f)_\infty$ which by (1) is the same as the degree of $(f)_0$ which is by definition the order of f . On the other hand, the order of $f - a$ is the same as the order of g which by definition is the degree of $(g)_0$ which by (1) applied to g is the same as the order of $(g)_\infty$. This is (3). □

In particular note that a rational function of order n defines a bijection from $\mathbb{P}^1$ to $\mathbb{P}^1$ if and only if $n = 1$.

Definition 4.3. *A rational function of order one is called a Möbius transformation.*

Note that the general function Möbius transformation has the form

$$z \longrightarrow \frac{az + b}{cz + d},$$

where a, b, c and d are complex numbers. Of course we require that $az + b$ and $cz + d$ don't have the same root, that is, we require that $cz + d$ is not a scalar multiple of $az + b$.

On the other hand it is proved in a course on Galois Theory that the group of field automorphisms of $K(x)$, where K is field and x is a transcendental element, is also of the same form, that is, the automorphism group consists of all transformations of the form

$$x \longrightarrow \frac{ax + b}{cx + d}.$$

In terms of $\mathbb{P}^1$ note that any element of $\mathrm{GL}_2(\mathbb{C})$ acts naturally on $\mathbb{C}^2$, whence on $\mathbb{P}^1$. On the other hand any scalar matrix acts trivially on $\mathbb{P}^1$.

Definition 4.4. $\mathrm{PGL}_2(\mathbb{C})$ *denotes the quotient of* $\mathrm{GL}_2(\mathbb{C})$ *by the subgroup of scalar matrices.*

Thus a general element of $\mathrm{PGL}_2(\mathbb{C})$ is of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

an equivalence class of matrices.

It is also easy to write down the action of $\mathrm{PGL}_2(\mathbb{C})$. Indeed

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} aX + bY \\ cX + dY \end{bmatrix}.$$

On the other hand $[X : Y] = [X/Y : 1] = [z : 1]$, at least when $Y \neq 0$ and

$$\begin{aligned}[aX + bY : cX + dY] &= [(aX + bY)/(cX + dY) : 1] \\ &= [(a(X/Y) + b)/(c(X/Y) + d) : 1] \\ &= [(az + b)/(cz + d) : 1].\end{aligned}$$

Note that the condition that $cz + d$ is not a scalar multiple of $az + b$ is exactly the condition that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible. Finally note that the equality of transformations

$$z \longrightarrow \frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az + b}{cz + d}.$$

Putting all this together we have:

Proposition 4.5. *The following groups*

(1) the Möbius transformations
(2) automorphisms of the field $\mathbb{C}(z)$*, and*
(3) $\mathrm{PGL}_2(\mathbb{C})$

are isomorphic.