8. More about Möbius Transformations

Recall that a Möbius Transformations is a rational function of degree one, so that as a transformation $f$ of the extended complex plane

$$z \mapsto \frac{az + b}{cz + d}.$$  

Here $f(\infty) = \frac{a}{c}$ and $f(-d/c) = \infty$. Möbius Transformations are sometimes called linear transformations, for obvious reasons.

It is not hard to prove that the group of Möbius Transformations is exactly thrice transitive.

Indeed if $z_1, z_2, z_3$ are three points of $\mathbb{C}$, then

$$z \mapsto \frac{z_3 - z_2 z - z_1}{z_3 - z_1 z - z_2}$$

sends $z_1$ to zero, $z_2$ to infinity and $z_3$ to 1.

On the other hand, by direct computation, it is not hard to show that the only Möbius Transformation which fixes 0, 1 and $\infty$ is the identity transformation.

Using this fact, one can define an extremely important invariant of four ordered points $z_1, z_2, z_3$ and $z_4$ in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

**Definition 8.1.** Let $z_1, z_2, z_3$ and $z_4$ be four distinct points of the extended complex plane.

The **cross-ratio** of these four points is

$$\frac{z_2 - z_4 z_1 - z_3}{z_2 - z_3 z_1 - z_4}.$$  

By what we have just said, the cross-ratio $\lambda$ is the image of $z_1$ when you use a Möbius transformation to send $z_2$ to 1, $z_3$ to 0 and $z_4$ to $\infty$. Note the cross-ratio is invariant under the action of the Möbius Transformations and vice-versa:

**Lemma 8.2.** Given a pair of four distinct ordered points $z_1, z_2, z_3$ and $z_4$, and $z_1', z_2', z_3'$ and $z_4'$ of the extended complex plane, we may find a Möbius Transformations carrying one set to the other if and only if the cross-ratios $\lambda$ and $\lambda'$ are equal.

On the other hand the cross-ratio is not an invariant under changing the order:

**Lemma 8.3.** The group of Möbius Transformations that preserve the set $\{0, 1, \infty\}$ is isomorphic to $S_3$, generated by

$$z \mapsto 1/z \quad \text{and} \quad z \mapsto 1 - z.$$
Proof. Since the group of Möbius Transformations is precisely thrice transitive, the subgroup of Möbius Transformations that preserve the set \( \{0, 1, \infty\} \) is isomorphic to \( S_3 \).

On the other hand
\[
z \mapsto 1/z \quad \text{and} \quad z \mapsto 1 - z
\]
map to two different transpositions, and any two transpositions generate \( S_3 \).

The stabiliser of the set \( \{z_1, z_2, z_3, z_4\} \) turns out to always contain the Klein VierrerGruppe \( V \). Indeed the transformation,
\[
z \mapsto \lambda/z
\]
switches 0 and \( \infty \) and 1 and \( \lambda \), and by symmetry we can get any other permutation which switches two pairs of elements. The quotient of \( S_4 \) by the subgroup \( V \) is \( S_3 \).

**Definition-Lemma 8.4.** Given four points \( z_1, z_2, z_3 \) and \( z_4 \) in the plane, the \( j \)-invariant is the number
\[
2^8 \left( \frac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda - 1)^2} \right)^3,
\]
where \( \lambda \) is the cross-ratio.

Two sets of four points in \( \mathbb{P}^1 \) may be mapped to each other by an Möbius transformation of \( \mathbb{P}^1 \) if and only if they have the same \( j \)-invariant.

**Proof.** We first check that the \( j \)-invariant does not depend on the order.

It suffices to check what happens under the two transpositions. This is an easy check.

Now use a little Galois theory. We want the fixed field \( K \) of \( S_3 \) acting on \( L = \mathbb{C}(\lambda) \). Then \( L/K \) is Galois, and the degree is six.

Now \( E = \mathbb{C}(j) \subset K \) and \( \lambda \) satisfies a degree six polynomial over \( E \).

Thus \( L/E \) has degree at most six. But then \( K = E \). \( \square \)

It is interesting to note that there are two very special values of the \( j \)-invariant, 0 and 1728. They correspond to the sets of four points which have extra automorphisms. In fact, one set corresponds to \( 0, 1, \infty, -1 \).

Here we have an extra \( \mathbb{Z}_2 \), corresponding to
\[
z \mapsto 1/z.
\]
The other corresponds to \( 1, \omega, \omega^2, 0 \), where \( \omega \) is a cube root of unity.

Here we have an extra \( \mathbb{Z}_3 \), corresponding to
\[
z \mapsto \omega z.
\]