9. Circles and lines

Back to the cross-ratio. Suppose we have z_1, z_2, z_3, z_4. I claim that the cross-ratio is real if and only if these four points lie on a circle.

Indeed the cross-ratio is equal to

$$\frac{z_2 - z_4}{z_2 - z_3}\frac{z_1 - z_3}{z_1 - z_4}.$$

Taking arguments, the cross-ratio is real if and only if the difference

$$\arg(\frac{z_1 - z_3}{z_1 - z_4}) - \arg(\frac{z_2 - z_3}{z_2 - z_4})$$

is 0 or $\pm\pi$.

The first angle is the angle $z_4 z_1 z_3$ and the second angle is the angle $z_3 z_2 z_4$. On the other hand, some classical Greek geometry says that if z_3 and z_4 are two points on a circle C then the angle subtended by z_3 and z_4 to two points on the circle C is always either the same or supplementary.

Another way to see that the cross-ratio is real, is to observe that it is enough to prove that the image of the real axis under a Möbius Transformation is a circle. Suppose that the transformation in question is the *inverse* of

$$w \longrightarrow \frac{aw + b}{cw + d} = z.$$

As z is real, we have

$$\frac{aw + b}{cw + d} = \frac{\bar{a}\bar{w} + \bar{b}}{\bar{c}\bar{w} + \bar{d}}.$$

If one multiplies out we get

$$(a\bar{c} - c\bar{a})|w^2| + (a\bar{d} - c\bar{b})w + (b\bar{c}_d\bar{a})w + b\bar{d} - d\bar{b} = 0$$

The coefficient of x^2 and y^2 is the same, so either we get a straight line or a circle. If $a\bar{c} - c\bar{a} = 0$ then this is the equation of a straight line. Otherwise if we divide through by $a\bar{c} - c\bar{a} = 0$ and complete the square we get:

$$\left| w + \frac{\bar{a}d - \bar{c}b}{\bar{a}c - \bar{c}a} \right| = \left| \frac{ad - bc}{\bar{a}c - \bar{c}a} \right|.$$

Note that the equation of a circle, with centre z_0 and radius ρ can be written as

$$|z - z_0| = \rho.$$

Definition 9.1. *Given two points z and z^*, we will say that these points are* ***inverse with respect to the circle*** *(aka symmetric with respect to the circle), if they are collinear with the centre, the product* *of their distances to the centre is ρ^2 and they are on the same side of the circle.*

If $p = z_0 + re^{i\theta}$ and the circle is

$$|z - z_0| = \rho.$$

then the inverse of p is $q = z_0 + (\rho^2/r)e^{i\theta}$.

Let (z_1, z_2, z_3, z_4) denote the cross-ratio of z_1, z_2, z_3 and z_4.

Theorem 9.2. *Suppose that z_1, z_2 and z_3 are three points of the circle C. Then z and z^* are inverse points if and only if*

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}.$$

In particular if the circle C is the image of the real axis under a linear transformation then inverse points with respect to C correspond to complex conjugate points.

Proof. Note that the cross-ratio is invariant under linear transformations. We have

$$\begin{aligned}
\overline{(z, z_1, z_2, z_3)} &= \overline{(z - z_0, z_1 - z_0, z_2 - z_0, z_3 - z_0)} \\
&= (\bar{z} - \bar{z}_0, \frac{\rho^2}{z_1 - z_0}, \frac{\rho^2}{z_2 - z_0}, \frac{\rho^2}{z_3 - z_0}) \\
&= (\frac{\rho^2}{\bar{z} - \bar{z}_0}, z_1 - z_0, z_2 - z_0, z_3 - z_0) \\
&= (\frac{\rho^2}{\bar{z} - \bar{z}_0} + z_0, z_1, z_2, z_3) \\
&= (z^*, z_1, z_2, z_3),
\end{aligned}$$

where we applied the transformation $z \longrightarrow z - z_0$ to get the first line, we used the fact that for a point on the circle,

$$\overline{z' - z_0} = \frac{\rho^2}{z' - z_0}$$

to get from the first line to the second line, we applied the transformation

$$z \longrightarrow \frac{\rho^2}{z - z_0}$$

which is its own inverse, to get from the second line to the third line and we applied the transformation $zz + z_0 \longrightarrow$ to get from the third line to the fourth line ☐

Corollary 9.3. *Let C be a circle and let z and z^* be inverse points with respect to the cicle.*

Suppose that the Möbius transformation $f(x)$ carries the circle C to the circle C'. Then the images w and w^* are inverse points with respect to the circle C'.

Informaly, a Möbius transformation carries inverse points to inverse points. This result follows from the fact that if C is the image of the real axis by some linear transformation, then inverse points with respect to C correspond to complex conjugate points.

Now suppose that $z = z_0 + \rho e^{i\phi}$ is a point of the circle. Then

$$\begin{aligned}\frac{|z-p|}{|z-q|} &= \frac{|\rho e^{i\phi} - re^{i\theta}|}{|\rho e^{i\phi} - \rho^2/re^{i\theta}|} \\ &= \frac{\rho}{r}\frac{|\rho e^{i\phi} - re^{i\theta}|}{|re^{i\phi} - \rho e^{i\theta}|} \\ &= \frac{\rho}{r}\frac{|\rho e^{i\phi-\theta} - r|}{|r - \rho e^{i\theta-\phi}|} \\ &= \frac{\rho}{r}.\end{aligned}$$

This gives us another way to write down the equation of a circle.

$$\frac{|z-p|}{|z-q|} = k,$$

where k is a positive real constant, is the equation of a circle, such that p and q are inverse points of this circle. Indeed, we just need to find a circle, such that the points p and q are inverse, where

$$k = \frac{\rho}{r}.$$

The points on the circle are a fixed ratio from two points p and q; classically these are known as the circles of Apollonius.

In fact, there is another way to look at all of this. Suppose one looks at the linear transformation,

$$w = k\frac{z-a}{z-b}.$$

One way to study any transformation is to investigate what happens to circles and lines. Now if one starts with a circle of radius ρ, centre the origin, then the image of this circle has the equation

$$\frac{|z-a|}{|z-b|} = \frac{\rho}{|k|},$$

a circle of Apollonius.

It is interesting to look at this a little more. Given the two points a and b, one can also look at the one parameter family of circles that pass through a and b. Thus we get two one dimensional families of circles, determined by a and b. The circles with inverse point a and b and the circles containing these two points. This is called a **Steiner system**, or **circular net**. This pair of families of circles enjoy many interesting properties. For example:

(1) There is exactly one circle from each family through any point of the plane, other than either a or b.
(2) Every circle from one system meets a circle of the other at right angles.
(3) Reflection in a circle from one family, permutes the circles of the same family and fixes the circles of the other family.

Let's consider one seemingly special case, when when $a = 0$ and $b = \infty$. Circles containing both a and b are lines through the origin. Circles with these points as inverse points are circles centred at the origin.

Suppose that $c \in \mathbb{C}^*$. There is exactly one line through the origin containing c and there is exactly one circle centred at the origin containing c.

A line through the origin is certainly orthogonal to a circle centred at the origin.

Reflection in a line through the origin certainly fixes a circle through the origin, and it certainly sends a line through the origin to another line through the origin. Similarly reflection in a circle centred at the origin fixes a line through the origin and sends a circle centred at the origin to a circle centred at the origin.

On the other hand, there is a Möbius transformation which carries two arbitrary points a and b to 0 and ∞ and this Möbius transformation preserves all three properties.