MODEL ANSWERS TO THE THIRD HOMEWORK

1. This is done by a trick. First suppose that $m = 1$. Then we get

$$\frac{1}{1-z} = 1 + z + z^2 + \dots,$$

simply because we recognize

$$\frac{1}{1-z}$$

as a geometric series with the given sum. Now suppose that we know the power series expansion of $(1-z)^{-m}$. The derivative of this is

$$-m(1-z)^{-m-1}.$$

On the other hand, we can differentiate a series term by term. For example,

$$(1-z)^{-2} = -\frac{1}{2}\left(1 + 2z + 3z^2 + 4z^3 + \dots\right).$$

More generally,

$$\begin{aligned}
(1-z)^{-(m+1)} &= -\frac{1}{m+1}\frac{d}{dz}(1-z)^{-m} \\
&= (-1)^{m-1}\frac{1}{(m+1)!}\frac{d^{m-1}}{dz^{m-1}}(1 + z + z^2 + \dots,) \\
&= (-1)^{m-1}\frac{1}{(m+1)!}\left(m! + (m+1)!z + \frac{(m+2)!}{2}z^2 + \dots + \frac{(m+k)!}{k!}z^k + \dots\right) \\
&= (-1)^{m-1}\frac{1}{m+1}\left(1 + (m+1)z + \frac{(m+2)(m+1)}{2}z^2 + \dots + \binom{m+k}{m}z^k + \dots\right)
\end{aligned}$$

2. Note that

$$\frac{2z+3}{z+1} = \frac{2y+5}{y+2},$$

where $y = z - 1$.

Now

$$\begin{aligned}\frac{1}{y+2} &= \frac{1}{2}\frac{1}{(1+y/2)} \\ &= \frac{1}{2}\left(1-\frac{y}{2}+\left(\frac{y}{2}\right)^2-\left(\frac{y}{2}\right)^3+\dots\right) \\ &= \frac{1}{2}\left(1-\frac{y}{2}+\frac{y^2}{4}+\dots+(-1)^n\frac{y^n}{2^n}+\dots\right) \\ &= \frac{1}{2}-\frac{y}{4}+\frac{y^2}{8}+\dots+(-1)^n\frac{y^n}{2^{n+1}}+\dots.\end{aligned}$$

Thus

$$\begin{aligned}\frac{2z+3}{z+1} &= \frac{2y+5}{y+2} \\ &= 2+\frac{1}{y+2} \\ &= \frac{5}{2}-\frac{y}{4}+\frac{y^2}{8}+\dots+(-1)^n\frac{y^n}{2^{n+1}}+\dots \\ &= \frac{5}{2}-\frac{(z-1)}{4}+\frac{(z-1)^2}{8}+\dots+(-1)^n\frac{(z-1)^n}{2^{n+1}}+\dots.\end{aligned}$$

The radius of convergence is 2, as the radius of convergence of

$$\frac{1}{(1+y/2)} = \frac{1}{(1+(z-1)/2)}$$

is 2.

3. By definition $\cos z$ and $\sin z$ are linear combinations of e^{iz} and e^{-iz}. Now

$$e^{ii} = e^{-1} = 1/e \qquad \text{and} \qquad e^{i(-i)} = e^1 = e.$$

Thus

$$\cos i = \frac{1}{2}\left(e+\frac{1}{e}\right)$$

and

$$\sin i = \frac{1}{2i}\left(e-\frac{1}{e}\right).$$

4. Suppose that $w = se^{i\phi} = 2^i$. Taking logs of both sides, we get

$$\log s + i\phi = i(\log 2 + 2k\pi i) = i\log 2 - 2k\pi,$$

for some integer k.

Equating real and imaginary parts, we get $\log s = -2k\pi$, whence $s = e^{-2k\pi}$ and $\phi = \log 2$. Thus

$$w = e^{-2k\pi} e^{i\log 2} = e^{2l\pi}(\cos(\log 2) + i\sin(\log 2)),$$

where $l = -k$ is any integer.
Now suppose that $w = i^i$. Taking logs of both sides, we get

$$\log s + i\phi = i(\log i + 2k\pi i) = i(i\pi/2 + 2k\pi i) = -(4k+1)\pi/2.$$

Thus, equating real and imaginary parts,

$$s = e^{-(4k+1)\pi/2} \qquad \text{and} \qquad \phi = 0.$$

So

$$w = e^{(4l+3)\pi/2},$$

where l is any integer.
Suppose that $w = (-1)^{2i}$. Taking logs of both sides, we get

$$\log s + i\phi = 2i(\log 1 + (2k+1)\pi i) = -(4k+2)\pi,$$

for some integer k. Thus, equating real and imaginary parts,

$$s = e^{-(4k+2)\pi} \qquad \text{and} \qquad \phi = 0.$$

So

$$w = e^{(4l+2)\pi},$$

where l is any integer.
5. Suppose that $w = z^z$. Then $w = se^{i\phi}$ and $z = re^{i\theta}$ for appropriate r, s, θ and ϕ.
Now take logs of both sides

$$\begin{aligned}\log s + i\phi &= \log w \\ &= z\log z \\ &= re^{i\theta}(\log r + i(2\pi k + \theta)) \\ &= (r\log r)e^{i\theta} + i(2\pi k + \theta)re^{i\theta}.\end{aligned}$$

Now the complex conjugate of the last expression is

$$(r\log r)e^{-i\theta} - i(2\pi k + \theta)re^{-i\theta}.$$

Thus the real part is

$$\log s = r\log r\cos\theta - r(2\pi k + \theta)\sin\theta$$

and the imaginary part is

$$\phi = r\log r\sin\theta + r(2\pi k + \theta)\cos\theta$$

So

$$s = \exp(r\log r\cos\theta - r(2\pi k + \theta)\sin\theta).$$

Thus the real part of z^z is
$\exp\left(r\log r\cos\theta - r(2\pi k+\theta)\sin\theta\right)\cos\left(r\log r\sin\theta + r(2\pi k+\theta)\cos\theta\right)$
and the imaginary part of z^z is
$\exp\left(r\log r\cos\theta - r(2\pi k+\theta)\sin\theta\right)\sin\left(r\log r\sin\theta + r(2\pi k+\theta)\cos\theta\right).$