Here are some practice problems for the first midterm culled from various locations (several problems are a bit more involved than the midterm problems but are hopefully useful for review):

1. Let \( f: U \rightarrow \mathbb{C} \) be a holomorphic function. If \( z_0 \in U \) is a point such that \( f'(z_0) \neq 0 \) then show that \( f \) preserves angles between smooth curves intersecting at \( z_0 \).

Find a biholomorphic map between the two regions \( U \) and \( V \), where \( U \) is the second quadrant of the unit disc,
\[
U = \{ z \in \mathbb{C} \mid |z| < 1, \pi/2 < \arg(z) < \pi \}
\]
and \( V \) is the area outside the unit disc of the first quadrant:
\[
V = \{ z \in \mathbb{C} \mid |z| > 1, 0 < \arg(z) < \pi/2 \}.
\]

2. Let \( f(z) \) be an entire function. State Cauchy’s integral formula, relating the \( n \)th derivative of \( f \) at a point \( a \) with the values of \( f \) on some circle around \( a \).
State Liouville’s theorem, and deduce it from Cauchy’s integral formula.
Suppose that for some \( k \) we have that \( |f(z)| \leq |z|^k \) for all \( z \). Prove that \( f \) is a polynomial.

3. What is the radius of convergence of the power series
\[
\sum_{n=1}^{\infty} \frac{z^n}{n}.
\]

4. Find a conformal transformation \( f(z) \) that maps the region
\[
U = \{ z \in \mathbb{C} \mid 0 < \arg(z) < \frac{3\pi}{2} \}
\]
on to the strip
\[
V = \{ z \in \mathbb{C} \mid 0 < \text{Im}(z) < 1 \}.
\]
Hence find a bounded harmonic function \( \phi \) on \( U \) subject to the boundary conditions \( \phi = 0 \) on \( \arg z = 0 \) and \( \phi = A \) on \( \arg z = 3\pi/2 \) for some real constant \( A \).

5. Using Cauchy’s integral formula, write down the value of a holomorphic function \( f(z) \) where \( |z| < 1 \) in terms of a contour integral around the unit circle, \( \zeta = e^{i\theta} \).
By considering the point $1/z$ show that
\[ f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} \, d\theta. \]

By setting $z = re^{i\alpha}$, show that for any harmonic function $u(r, \alpha)$,
\[ u(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} u(1, \theta) \frac{1 - r^2}{1 - 2r \cos(\alpha - \theta) + r^2} \, d\theta. \]
Assuming that the harmonic conjugate $v(r, \theta)$ can be written as
\[ v(r, \alpha) = v(0) + \frac{1}{\pi} \int_0^{2\pi} u(1, \theta) \frac{r \sin(\alpha - \theta)}{1 - 2r \cos(\alpha - \theta) + r^2} \, d\theta, \]
deduce that
\[ f(z) = iv(0) + \frac{1}{2\pi} \int_0^{2\pi} u(1, \theta) \frac{\zeta + z}{\zeta - z} \, d\theta. \]

6. Let $U$ be the disc centred at $a$ with radius $r$ and let $f : U \to \mathbb{C}$ be a holomorphic function. Using Cauchy’s integral formula, show that for every $0 < s < r$,
\[ f(a) = \int_0^1 f(a + se^{2\pi it}) \, dt. \]
Deduce that if
\[ |f(z)| \leq |f(a)| \quad \text{for every} \quad z \in U, \]
then $f$ is constant.
Now specialise to the case when $a = 0$ and $r = 1$, so that $U$ is the unit disc. If $f(0) = 0$ and
\[ f : U \to U \]
then show that
\[ |f(z)| \leq |z| \quad \text{for every} \quad z \in U. \]
Moreover if $|f(w)| = |w|$ for some $w \neq 0$ then there exists $\lambda$ with $|\lambda| = 1$ such that
\[ f(z) = \lambda z \quad \text{for every} \quad z \in U. \]