1. Classification of finitely generated field extensions

The following problem will be the main focus of these lectures:

Problem 1.1. *Fix a field K.*

Classify all finitely generated field extensions L/K.

We will be mostly interested in the case when $K = \mathbb{C}$, which is not to say that either the case when K is not algebraically closed or the case when K has characeristic zero is not interesting.

Let n be the transcendence degree of L/K. The complexity of (1.1) increases as n increases. If $n = 0$ then this is essentially the subject of Galois theory (in this case, to make things interesting, we would relax the condition that K is algebraically closed). When $n > 0$ it is necessary to think about the whole problem in a completely different way. Recall that a rational map $\phi\colon X \dashrightarrow Y$ between two quasi-projective varieties is a morphism $f\colon U \longrightarrow Y$ defined on some open subset U of X.

Definition 1.2. *Let $X \subset \mathbb{P}^n$ be a quasi-projective variety over the field K. The **function field** $K(X)/K$ of X is the set of all rational functions $\phi\colon X \dashrightarrow \mathbb{A}^1$.*

Perhaps the most direct way to compute the function field of X is to pick any open affine subset U of X. Then $K(X)/K$ is simply the field of fractions of the coordinate ring $A(U)$.

Lemma 1.3. *Let L/K be a finitely generated field extension, where K has characteristic zero. Suppose either that K is algebraically closed or L/K is not finite.*

Then we may find a quasi-projective variety X such that the field extension $K(X)/K$ is isomorphic to L/K.

Proof. We may find n algebraically independent elements $x_1, x_2, \dots, x_n$ of L. Let $M = K(x_1, x_2, \dots, x_n)$. Then M/K is a purely transcendental extension and L/M is a finite extension. By the theorem of the primitive element (here is where we use the fact that the characteristic is zero) there is an element $x_{n+1} \in L$ such that $L = M(x_{n+1})$. Let $m(x) \in M[x]$ be the minimum polynomial of x_{n+1}. Then the coefficients of m are rational functions of $x_1, x_2, \dots, x_n$ with coefficients in K. Clearing denominators, it follows that we can find a polynomial

$$f(y_1, y_2, \dots, y_{n+1}) \in K[y_1, y_2, \dots, y_{n+1}] \qquad \text{such that} \qquad f(x_1, x_2, \dots, x_{n+1}) = 0.$$

Let $X \subset \mathbb{A}^{n+1}$ be the affine variety defined by f. Then the coordinate ring of X is isomorphic to

$$\frac{K[y_1, y_2, \dots, y_{n+1}]}{\langle f(y_1, y_2, \dots, y_{n+1}) \rangle}.$$

It follows that the function field of X is isomorphic to L. □

Note that one can compose dominant rational maps, so that there is a category of quasi-projective varieties and dominant rational maps.

Definition-Lemma 1.4. *Let $\phi \colon X \dashrightarrow Y$ be a dominant rational map between two quasi-projective varieties.*

Then there is a natural ring homorphism $\phi^* \colon K(Y) \longrightarrow K(X)$, which fixes K.

Proof. If $f \colon Y \dashrightarrow \mathbb{A}^1$ is a rational function then let $\phi^*(f) = f \circ \phi \colon X \dashrightarrow \mathbb{A}^1$. The rest is clear. □

Theorem 1.5. *There is an equivalence of categories between the category of quasi-projective varieties and dominant rational maps over K and the category of finitely generated field extensions of K.*

Proof. Define a functor by sending a quasi-projective variety X over K to the field extension $K(X)/K$ and sending a dominant rational map $\phi \colon X \dashrightarrow Y$ to the ring homomorphism $\phi^* \colon K(Y) \dashrightarrow K(X)$. We have to show that this functor is fully faithful (that is, the morphisms in the two categories are the same) and that this functor is essentially surjective. (1.3) says precisely that this functor is essentially surjective. To show that the functor is fully faithful it suffices to show that given a ring homomorphism $f \colon K(Y) \longrightarrow K(X)$ there is a rational map $\phi \colon X \dashrightarrow Y$ such that $f = \phi^*$. We may assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are both affine. Suppose that coordinates on $\mathbb{A}^m$ are $y_1, y_2, \dots, y_m$. Let $q_i = f(y_i)$, so that q_i are rational functions of $x_1, x_2, \dots, x_n$, coordinates on $\mathbb{A}^n$. Define a rational map

$$\phi \colon X \dashrightarrow \mathbb{A}^m,$$

by the rule

$$(x_1, x_2, \dots, x_n) \longrightarrow (q_1, q_2, \dots, q_m).$$

It is easy to check that $\phi^* = f$. □

By virtue of (1.5) we may restate (1.1) as:

Problem 1.6. *Classify quasi-projective varieties up to birational equivalence.*

Note that the transcendence degree n of $L = K(X)/K$ is precisely the dimension of X. Note also that (1.3) proves a much stronger result than stated. In fact every non-trivial field extension L/K can be realised as the function field of an affine hypersurface. However this is not the right way to look at all of this.

Lemma 1.7. *Let $\phi\colon C \dashrightarrow Y$ be a rational map, where C is a smooth curve and $Y \subset \mathbb{P}^n$ is a projective variety.*

Then ϕ extends to a morphism $\phi\colon C \longrightarrow Y$.

Proof. By assumption we are given a morphism $\phi\colon U \longrightarrow Y$ defined on a dense open subset U of C. Since $Y \subset \mathbb{P}^n$ is closed we may assume that $Y = \mathbb{P}^n$. The complement of U is a finite set of points $p_1, p_2, \dots, p_k$. Since this result is local about any one of these points, we may assume that $U = C - \{p\}$.

We give a proof of this fact which is only valid over $\mathbb{C}$. It is easy to adapt this proof to the general case using the language of DVR's. Working analytically locally, we may assume that $C = \Delta$ and p is the origin. By assumption ϕ is given locally by a collection of meromorphic functions f_i,

$$z \longrightarrow [f_0(z) : f_1(z) : \dots : f_n(z)].$$

Now each meromorphic function $f_i(z)$ has a Laurent expansion, $\sum a_j z^j$, so that we may write

$$f_i(z) = z^{m_i} g_i(z),$$

where each $m_i \in \mathbb{Z}$ and $g_i(z)$ is a holomorphic function. Let

$$m = \min_i(m_0, m_1, \dots, m_n).$$

Then ϕ is equally well given by

$$z \longrightarrow [f_0(z) : f_1(z) : \dots : f_n(z)] = [z^{-m} f_0(z) : z^{-m} f_1(z) : \dots : z^{-m} f_n(z)].$$

By our choice of m each

$$z^{-m} f_i(z),$$

is holomorphic and one of them is non-zero at the origin. □

Using (1.7) and (1.5) it is possible to restate (1.1) once again, in the case when the transcendence degree $n = 1$ is one. The classification of field extensions of transcendence degree one is equivalent to the classification of smooth curves. By virtue of (1.7) the issue of rational maps which are not morphisms does not play any role.

So now let us turn our attention to the study of smooth curves:

Definition 1.8. *Let C be a projective curve. The* ***arithmetic genus*** *p_a of C is the dimension $h^1(C, \mathcal{O}_C)$ of $H^1(C, \mathcal{O}_C)$. The* ***geometric genus*** *is the arithmetic genus of the normalisation.*

Without going into too many details (one can easily fill a whole course on the topic of the classification of smooth curves) unfortunately the picture is quite bad. The point is that there is a natural parameter space for the set of all curves of genus g, called the **moduli space of curves of genus** g, $\mathcal{M}_g$, which is a quasi-projective variety. $\mathcal{M}_g$ is natural in the following sense. The points of $\mathcal{M}_g$ are in bijection with the set of isomorphism classes of smooth curves of genus g. Let

$$U = \{\, [C] \in \mathcal{M}_g \,|\, C \text{ has only the trivial automorphism} \,\},$$

be the set of points $[C] \in \mathcal{M}_g$ corresponding to curves C with no automorphisms. Then U is a dense open subset of $\mathcal{M}_g$, for $g \geq 3$ and there is a universal curve over U, $\pi\colon \mathcal{C} \dashrightarrow U$, whose fibre $\pi^{-1}([C])$ over the point $[C]$ is isomorphic to the curve C. The unfortunate aspect of all of this is that U and hence $\mathcal{M}_g$ has dimension $3g-3$. A moment's thought will convince the reader that this means that the classification of smooth curves is quite hard. In fact, in a sense we will make precise later on, the general curve of genus g is essentially unknowable.