11. The log discrepancy

Definition 11.1. *Let (X, Δ) be a log pair. If $\pi\colon Y \longrightarrow X$ is any birational morphism such that $K_Y + \Gamma$ is $\mathbb{Q}$-Cartier, and $E_1, E_2, \dots, E_k$ are the exceptional divisors, then we may write*

$$K_Y + \Gamma = K_X + \pi_*^{-1}\Delta + E = \pi^*(K_X + \Delta) + \sum a_i E_i,$$

for rational numbers $a_1, a_2, \dots, a_k$, where $\pi_*^{-1}\Delta$ is the strict transform of Δ and $E = \sum E_i$ is the sum of the exceptional divisors. The number $a_i = a(E_i, X, \Delta)$ is called the ***log discrepancy*** *of the divisor E_i.*

The ***log discrepancy*** *$a = a(X, \Delta)$ of (X, Δ) is the infimum of the log discrepancies over all exceptional divisors of all birational morphisms.*

Note that it is not necessary to assume that $\Delta \geq 0$ to define the log discrepancy. We only need that X is normal and $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

We run through one computation of the log discrepancy. Let X be the cone over a rational normal curve of degree d. If we blow up $\pi\colon Y \longrightarrow X$ the vertex of the cone then π is a log resolution and the exceptional divisor E is a copy of $\mathbb{P}^1$; $E^2 = -d$. We may write

$$K_T + E = \pi^* K_S + aE,$$

for some rational number a. If we do both sides with respect to E we get

$$-2 = \deg K_{\mathbb{P}^1} = \deg K_E = (K_T + E) \cdot E = \pi^* K_S \cdot E + aE^2 = -ad.$$

Thus

$$a = \frac{2}{d}.$$

Definition 11.2. *Let K/k be a field extension. A* ***valuation*** *ν of K/k is a map*

$$\nu\colon K \longrightarrow \mathbb{Z} \cup \{\infty\},$$

such that

(1) *$\nu(f) = \infty$ if and only if $f = 0$.*
(2) *$\nu(fg) = \nu(f) + \nu(g)$.*
(3) *$\nu(f + g) \geq \max(\nu(f), \nu(g))$.*
(4) *$\nu(k^*) = \{0\}$.*

Example 11.3. *Let X be a normal projective variety and let $D \subset X$ be a prime divisor. Then the order of vanishing of a rational function along D determines a valuation,*

$$\nu_D(f) = \operatorname{mult}_f D.$$

*If ν is a valuation such that $\nu = \nu_E$ for some divisor E, possibly exceptional, then we will call ν an **algebraic valuation**. The **centre** of ν is the image of E in X.*

The language of valuations provides a convenient way to refer to the same exceptional divisors, on different models. Note that if $E_1 \subset Y_1$ and $E_2 \subset Y_2$ are two divisors on birational varieties Y_1 and Y_2, then $\nu_{E_1} = \nu_{E_2}$ if and only if there is a birational map $\phi\colon Y_1 \dashrightarrow Y_2$ which is an isomorphism in a neighbourhood of the generic points of E_1 and E_2.

The log discrepancy is a birational invariant, in the following weak sense:

Lemma 11.4. *Let (X, Δ) be a log pair and let ν be a valuation.*

The log discrepancy only depends on ν.

Proof. Suppose that we are given $\pi_i\colon Y_i \longrightarrow X$ two birational morphisms on which the centre of ν_i is a divisor E_i. If $\phi\colon Y_1 \dashrightarrow Y_2$ is the induced birational map then ϕ is an isomorphism at the generic point of E_1. We may write

$$K_{Y_i} + \Gamma_i = \pi_i^*(K_X + \Delta) + a_i E_i + J_i,$$

where J_i does not involve E_i and we want to show that $a_1 = a_2$. Pick a meromorphic differential form ω_2 on Y_2 and let $\omega_1 = \phi^*\omega_2$. Then

$$a_i = 1 - \operatorname{mult}_{E_i} \pi_i^*(K_X + \Delta) + \operatorname{mult}_{E_i} \omega_i,$$

which is independent of i by construction. □

Definition 11.5. *We say that a log pair (X, Δ) is **canonical** if the log discrepancy is at least one.*

Lemma 11.6. *Let $\phi\colon X \dashrightarrow Y$ be a birational map between two projective varieties with canonical singularities and let m be a positive integer, such that both mK_X and mK_Y are Cartier.*

Then there is a natural isomorphism

$$H^0(X, \mathcal{O}_X(mK_X)) \simeq H^0(Y, \mathcal{O}_X(mK_Y)).$$

Proof. Let $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ be a common resolution of ϕ. Then we just have to prove the result for p and q. Replacing ϕ by p we may assume that ϕ is a morphism, a log resolution of X.

Let V the indeterminancy locus of ϕ^{-1}. Suppose that ω is a pluricanonical form on X. Then $\eta = \phi_*\omega$ is a rational form on Y whose poles are concentrated on V, which is a closed subset of codimension at least two. But then η is in fact regular. Thus there is a natural map

$$H^0(X, \mathcal{O}_X(mK_X)) \longrightarrow H^0(Y, \mathcal{O}_Y(mK_Y)).$$

Conversely suppose that η is a pluricanonical form on Y. By assumption,

$$K_X = \pi^* K_Y + E,$$

where $E \geq 0$ is exceptional. Then

$$\begin{aligned} \pi^*\eta &\in H^0(X, \mathcal{O}_X(m\pi^* K_Y)) \\ &\subset H^0(X, \mathcal{O}_X(m\pi^* K_Y + mE)) \\ &= H^0(X, \mathcal{O}_X(mK_X)). \end{aligned}$$

□

Lemma 11.7. *Let $\pi\colon X \longrightarrow Y$ blow up a smooth variety V of codimension k, with exceptional divisor E.*

Then the log discrepancy of E is equal to k.

Proof. We have

$$K_X + E = \pi^* K_Y + aE,$$

where a is the log discrepancy. Restricting to E, we have

$$K_E = (K_X + E)|_E = \pi^* K_Y|_E + aE|_E = aE|_E.$$

Let F be a general fibre. Restricting to F, we have

$$-kH = K_{\mathbb{P}^{k-1}} = K_F = aE|_F = -aH,$$

where H is the class of a hyperplane. But then $a = k$. □

Lemma 11.8. *Let $(X, \Delta = \sum a_i \Delta_i)$ be a log smooth pair, where we allow some of the coefficients of Δ to be negative.*

If Δ has a component of coefficient greater than one, then set $a = -\infty$. Otherwise, let

$$a = \min_Z (k - \sum a_i),$$

where Z ranges over the irreducible components of the strata of the support of Δ, k is the codimension of Z and we sum over those components of Δ which contain Z.

Then the log discrepancy of $K_X + \Delta$ is a. In particular the log discrepancy of any pair is either at least zero, or it is $-\infty$ and if X is smooth and $\Delta = 0$ then X is canonical.

Proof. Suppose that Δ has a component C of coefficient $1 + \epsilon$, where $\epsilon > 0$. We are going to successively blow up X along a general smooth codimension two subset of C. Thus we might as well suppose that $S = X$ is a smooth surface and $\Delta = (1 + \epsilon)C$, where C is a smooth curve. Suppose that we blow up $\pi\colon T \longrightarrow S$ the point $p \in C$, with exceptional divisor E. As the log discrepancy of E with respect to K_S is 2, we have

$$K_T + E = \pi^* K_S + 2E,$$

where E is the exceptional divisor. Let D be the strict transform of C. As $\pi^*C = D + E$, it follows that

$$K_T + (1+\epsilon)D + E = \pi^*(K_S + (1+\epsilon)C) + (1-\epsilon)E.$$

The log discrepancy of E is then $1-\epsilon$. On the other hand,

$$K_T + (1+\epsilon)D + \epsilon E = \pi^*(K_S + (1+\epsilon)C).$$

Note that D and E are now two smooth curves, intersecting transversally at a smooth point, where D has coefficient $1+\epsilon$ and E has coefficient ϵ. Now suppose that we blow up the intersection of D and E on T. Mutatis mutandis, a similar calculation shows that the exceptional divisor E_1 has log discrepancy $1-2\epsilon$ with respect to $K_T+(1+\epsilon)D+\epsilon E$ and so also with respect to $K_S+(1+\epsilon)C$. Moreover now we have two smooth curves intersecting transversally at a point, one with coefficient 2ϵ the other with coefficient $1+\epsilon$. If we blow up the intersection point, then we get an exceptional divisor with log discrepancy $1-3\epsilon$ and so on. Continuing in this way we get exceptional divisors of log discrepancy $1-k\epsilon$, for all $k \geq 0$. Thus the log discrepancy is $-\infty$.

Now suppose that Δ is a boundary. If we blow up Z, with exceptional divisor E, then we have

$$K_Y + E = \pi^*K_X + kE,$$

since the log discrepancy is k. Since

$$\pi_*^{-1}\Delta + (\sum a_i)E = \pi^*\Delta,$$

it follows that E has log discrepancy $k - \sum a_i$ with respect to $K_X + \Delta$.

Finally suppose that ν is some algebraic valuation. By (10.6), we may realise ν by blowing up smooth centres which intersect the support of Δ transversally. If we rewrite the equation above as

$$K_Y + \pi_*^{-1}\Delta + (\sum a_i + 1 - k)E = \pi^*(K_X + \Delta),$$

and observe that

$$\sum a_i + 1 - k = (a_1-1)+\cdots+(a_i-1)+a_i+(a_{i+1}-1)+\cdots+(a_k-1) \leq \min a_i,$$

since we are assuming that $a_i \leq 1$, it is easy to see that the log discrepancy is computed after one blow up. □

Proof of (9.2). We will only show that the plurigenera are birational invariants; a similar argument applies to the irregularity $q(X)$. By (11.8) it follows that X and Y are canonical and we may apply (11.6). □