12. Classification of Surfaces

The key to the classification of surfaces is the behaviour of the canonical divisor.

Definition 12.1. *We say that a smooth projective surface is* ***minimal*** *if* K_S *is nef.*

Warning: This is not the classical definition of a minimal surface.

Definition 12.2. *Let* S *be a smooth projective surface. We say that a curve* $C \subset S$ *is a* -1***-curve*** *if*

$$K_S \cdot C = C^2 = -1.$$

Theorem 12.3 (Cone Theorem)**.** *Let* S *be a smooth projective surface.*

Then there are countably many extremal rays $R_1, R_2, \dots$ *of the closed cone of curves of* S *on which* K_S *is negative, such that*

$$\overline{\mathrm{NE}}(S) = \overline{\mathrm{NE}}(S)_{K_X \geq 0} + \sum R_i.$$

Further, if $R = R_i$ *is any one of these* K_S*-extremal rays then there is a birational morphism* $\pi \colon S \longrightarrow Z$ *which contracts a curve* C *if and only if* C *spans the ray* R*. There are three possibilities:*

1. *(1)* Z *is a point and* $S \simeq \mathbb{P}^2$.
2. *(2)* $\pi \colon S \longrightarrow Z$ *is a* $\mathbb{P}^1$*-bundle over a smooth curve* Z.
3. *(3)* $\pi \colon S \longrightarrow Z$ *is a birational morphism contracting a* -1*-curve* C*, where* Z *is a smooth surface.*

In particular the relative Picard number of π *is one, each extremal ray* R_i *is spanned by a rational curve and if* H *is any ample divisor, there are only finitely many extremal rays* R_i *such that* $(K_X+H)\cdot R < 0$.

Remark 12.4. *The last two statements are sometimes informally stated as saying that the closed cone of curves is locally rational polyhedral on the* K_S*-negative side of the cone.*

Theorem 12.5 (Castelnuovo)**.** *Let* S *be a smooth projective surface and let* $C \subset S$ *be a curve.*

Then C *is a* -1*-curve if and only if there is birational morphism* $\pi \colon S \longrightarrow T$*, which blows up a smooth point* $p \in T$*, with exceptional divisor* C.

Theorem 12.6 (Abundance)**.** *Let* S *be a smooth projective surface.*

Then K_S *is nef if and only if* K_S *is semiample.*

Theorem 12.7 (Kodaira-Enriques)**.** *Let* T *be a smooth projective surface, with invariants* $\kappa = \kappa(T)$*,* $p_g = p_g(T)$ *and* $q = q(T)$*. Then* T *is birational to a surface* S *which falls into the following list:*

$\kappa = -\infty$**:**

Ruled surface — *$S \simeq \mathbb{P}^1 \times B$ where B is a smooth curve of genus $q(S) = g(B)$.*

$\kappa = 0$**:**

Abelian surface — *$p_g = 1$, $q = 2$. $S \simeq \mathbb{C}^2/\Lambda$.*

Bielliptic — *$p_g = 0$, $q = 1$. There is a Galois cover of $\tilde{S} \longrightarrow S$ of order at most 12 such that $\tilde{S} \simeq E \times F$, where E and F are elliptic curves.*

K3 surface — *$p_g = 1$, $q = 0$.*

Enriques surface — *$p_g = 0$, $q = 0$. There is an étale cover $\tilde{S} \longrightarrow S$ of order two, such that $\tilde{S}$ is a K3-surface.*

$\kappa = 1$**:**

Elliptic fibration — *there is a contraction morphism $\pi \colon S \longrightarrow B$ with general fibre a smooth curve of genus one. $P_m(S) > 0$ for all m divisible by 12.*

$\kappa = 2$**:**

General type — *ϕ_m is birational for all $m \geq 5$. In particular $P_m(X) > 0$ for all $m \geq 5$.*

In particular $\kappa \geq 0$ if and only if $P_{12} \geq 0$.

Definition 12.8. *Let X be a normal projective variety, let D be a nef divisor and let E be any divisor. The* ***nef threshold*** *is the largest multiple of E we can add to D, whilst preserving the nef condition:*

$$\lambda = \sup\{\, t \in \mathbb{R} \mid D + tE \text{ is nef}\}.$$

Definition 12.9. *Let X be a projective scheme and let D be a nef divisor. The* ***numerical dimension*** *$\nu(X, D)$ of D is the largest positive integer such that $D^k \cdot H^{n-k} > 0$, where H is an ample divisor.*

Note that if D is semiample then $\kappa(D) = \nu(D)$. We will need the following easy:

Lemma 12.10. *Let X be a normal projective variety and let D be a nef $\mathbb{Q}$-Cartier divisor.*

(1) If $\nu(D) = 0$ then D is semiample if and only if $\kappa(D) = 0$.
(2) If $\nu(D) = 1$ then D is semiample if and only if $h^0(X, \mathcal{O}_X(mD)) \geq 2$, for some $m > 0$.

In particular if $\nu(D) \leq 1$ then D is semiample if and only if $\nu(D) = \kappa(D)$.

Proof. Suppose that $\nu(D) = 0$. Then D is numerically trivial and it is semiample if and only if it is torsion. As $\kappa(D) = 0$, $D \sim_{\mathbb{Q}} B \geq 0$ and since B is numerically trivial, in fact $B = 0$.

Suppose that $\nu(D) = 1$. Pick m so that $|mD|$ contains a pencil. We may as well assume that $m = 1$. We may decompose this linear system into mobile and fixed part:

$$|D| = |M| + F.$$

Let $B_i \in |D|$, $B_1 \neq B_2$. Then we may write $B_i = C_i + F$. By assumption D is not numerically trivial but

$$0 = D^2 \cdot H^{n-2} = (C_1 + F) \cdot D \cdot H^{n-2} \geq C_1 \cdot D \cdot H^{n-2}.$$

As C_1 moves we must have

$$C_1 \cdot C_2 \cdot H^{n-2} = 0 \qquad \text{and} \qquad C_1 \cdot F \cdot H^{n-2} = 0.$$

In particular $C_1 \cap C_2 = \varnothing$. Thus $|M|$ is base point free and we get a morphism $X \longrightarrow \mathbb{P}^1$. Let $f\colon X \longrightarrow \Sigma$ be the Stein factorisation. By assumption C_1 and C_2 are two different fibres. We have $C_1 \cap F = \varnothing$. Thus F is supported on the fibres of f. As it is Cartier and nef, it must be a multiple of a fibre. But then F is semiample and so D is semiample. □

Definition 12.11. *Let $\pi\colon X \longrightarrow U$ be a projective morphism.*

*The **relative cone of curves** is the cone generated by the classes of all curves contracted by π,*

$$\overline{\mathrm{NE}}(X/U) = \{\, \alpha \in \mathrm{NE}(X) \,|\, \pi_* \alpha = 0 \,\}.$$

*We say that a $\mathbb{Q}$-Cartier divisor H is **π-ample** (aka **relatively ample**, aka **ample over** U) if mH is relatively very ample (that is, there is an embedding i of X into $\mathbb{P}^n_U = \mathbb{P}^n \times U$ over U such that $\mathcal{O}_X(mH) = i^*\mathcal{O}(1)$).*

We say that an $\mathbb{R}$-divisor is relatively ample if and only if it is a positive linear combination of relatively ample $\mathbb{Q}$-divisors.

Note that if U is projective, then H is relatively ample if and only if there is an ample divisor G on U, such that $H + \pi^* G$ is ample. Note also that an $\mathbb{R}$-divisor is relatively ample if and only if it defines a positive linear functionall on $\overline{\mathrm{NE}}(X/U) - \{0\}$. Note that many of the definitions for $\mathbb{Q}$-divisors extend to $\mathbb{R}$-divisors. In particular, the property of being nef and the numerical dimension.

Proof of (12.3). Pick an extremal ray $R = \mathbb{R}^+\alpha$ of the closed cone of curves. We may pick an ample $\mathbb{R}$-divisor H such that

$$(K_S + H) \cdot \beta \geq 0,$$

for all $\beta \in \overline{\mathrm{NE}}(S)$ with equality if and only if $R = \mathbb{R}^+\beta$. In particular $D = K_S + H$ is a nef $\mathbb{R}$-Cartier divisor. The key technical point is to establish that R is rational, so that we may choose H to be an ample

$\mathbb{Q}$-divisor. In fact we will prove much more, we will prove that R is spanned by a curve. Let $\nu = \nu(S, D)$. There are three cases:

- $\nu = 0$,
- $\nu = 1$, and
- $\nu = 2$.

If $\nu = 0$, then $K_S + H$ is numerically trivial, and $-K_S$ is numerically equivalent to H. In other words $-K_S$ is ample. Moreover every curve C spans R. Thus S is a Fano surface of Picard number one and it follows that $S \simeq \mathbb{P}^2$. Note that R is rational in this case.

If $\nu = 1$, then we will defer the proof that R is rational. So assume that H is rational. We first prove that D is semiample. Consider asymptotic Riemann-Roch. $D^2 = 0$, by assumption.

$$D \cdot (-K_S) = D \cdot H > 0,$$

also by assumption. Thus $\chi(X, \mathcal{O}_X(mD))$ grows linearly. Since

$$h^2(S, \mathcal{O}_S(mD)) = h^0(S, \mathcal{O}_S(K_S - mD)) = 0,$$

for m sufficiently large, it follows that there is a positive integer $m > 0$ such that $|mD|$ contains a pencil. By (12.10) it follows that D is semiample. Let F by the general fibre of the corresponding morphism $\pi \colon S \longrightarrow C$. Then F is a smooth irreducible curve, $F^2 = 0$ and $-K_S \cdot F > 0$. By adjunction,

$$0 > (K_S + F) \cdot F = K_F = 2g - 2.$$

It follows that $g = 0$ so that $F \simeq \mathbb{P}^1$. Moreover since $R = \mathbb{R}^+[F]$ is extremal, the relative Picard number is one and so there are no reducible fibres. By direct classification it follows that there are no singular fibres. Thus π is a $\mathbb{P}^1$-bundle.

If $\nu = 2$ then D is big but not ample. As D is nef $D^2 > 0$. By continuity there is an ample $\mathbb{Q}$-divisor G such that $(K_S + G)^2 > 0$ and $(K_S + G) \cdot G > 0$, where $H - G$ is ample. Thus $K_S + G$ is big. By Kodaira's Lemma, $K_S + G \sim_{\mathbb{Q}} A + E$, where A is ample and $E \geq 0$. Now

$$(K_S + G) \cdot \alpha = D \cdot \alpha - (H - G) \cdot \alpha < 0.$$

On the other hand

$$0 > D \cdot \alpha = A \cdot \alpha + E \cdot \alpha > E \cdot \alpha.$$

It follows that α is spanned by a component of E so that $R = \mathbb{R}^+[C]$, for some component C of E. In particular R is rational and we may choose H to be a $\mathbb{Q}$-divisor. We have

$$0 > (K_S + C) \cdot C = K_C = 2g - 2.$$

Thus $g = 0$ and C is a -1-curve.

Replacing H by a multiple, we may assume that H is very ample (for the time being, we only need that it is Cartier) and $K_S + H$ is ample. Suppose that $H \cdot C = m > 0$. We may always assume that $m > 1$ (simply replace H by a multiple). If $G = H + (m-2)C$, then G is big and it is nef, since $G \cdot C = 2$. In particular G is ample by Nakai-Moishezon. Let $D = K_S + C + G$. Since we may write

$$D = (K_S + H) + C,$$

the stable base locus of D is contained in C. In particular for every curve $\Sigma \subset S$,

$$D \cdot \Sigma \geq 0,$$

with equality if and only if $\Sigma = C$. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_S(D - C) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_C(D) \longrightarrow 0.$$

Now $\mathcal{O}_C(D) = \mathcal{O}_C$, since $D|_C$ is a divisor of degree zero on $\mathbb{P}^1$. On the other hand,

$$H^1(S, \mathcal{O}_S(D - C)) = H^1(S, \mathcal{O}_S(K_S + G)) = 0,$$

by Kodaira vanishing. Thus there are no base points of D on C, so that D is semiample, and the resulting morphism $\pi \colon S \longrightarrow Z$ contracts C.

It remains to prove that Z is smooth. Consider the ample divisor $K_S + G$. We may always pick H very ample. In this case, I claim that $K_S + G$ is base point free (in fact it is very ample). The base locus is supported on C. Consider the exact

$$0 \longrightarrow \mathcal{O}_S(K_S + G - C) \longrightarrow \mathcal{O}_S(K_S + G) \longrightarrow \mathcal{O}_C(K_S + G) \longrightarrow 0.$$

Then $\mathcal{O}_C(K_S + G) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. As before,

$$H^1(S, \mathcal{O}_S(K_S + H + (m-3)C) = 0,$$

by Kodaira vanishing. Thus $K_S + G$ is base point free. Pick a general curve $\Sigma' \in |K_S + G|$. Then this must intersect C transversally at a single smooth point. But then $\Sigma = \pi_* \Sigma'$ is a smooth curve in Z. On the other hand $\Sigma + C \in |D|$. Since $|D|$ defines the contraction, the image of $\Sigma + C$, which is again Σ' is Cartier.

But any variety which contains a smooth Cartier divisor, is smooth in a neighbourhood of the divisor. Thus Z is smooth. $\square$