14. Bend and Break

In this section we will indicate how to prove an interesting consequence of Mori's famous bend and break result:

Theorem 14.1 (Mori-Miyaoka)**.** *Let X be a normal projective variety of dimension n, let H be a nef $\mathbb{R}$-divisor and let C a curve contained in the smooth locus of X.*

If $K_X \cdot C < 0$ then through every point $x \in C$ there passes a rational curve L_x such that

$$M \cdot L_x \leq 2n \frac{M \cdot C}{-K_X \cdot C}.$$

In fact we will only prove:

Theorem 14.2 (Mori)**.** *Let X be a smooth Fano variety of dimension n.*

Then X is covered by rational curves C such that $-K_X \cdot C \leq n+1$.

However once one sees the proof of (14.2) it is not hard to at least imagine how the proof of (14.1) goes. In both cases we start with a morphism $f\colon C \longrightarrow X$, such that $-K_X \underset{f}{\cdot} C < 0$. Pick a point $p \in C$ and let $x = f(p)$. The basic idea is to bend C, meaning that we will deform the morphism f, whilst preserving the condition that p maps to x. This will break C:

Lemma 14.3 (Rigidity-Lemma)**.** *Let $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Z$ be morphisms of varieties.*

If $f_*\mathcal{O}_X = \mathcal{O}_Y$, f is proper and there is a point $y \in Y$ such that the whole fibre $f^{-1}(y)$ is contracted to a point by g, then there is an open neighbourhood U of y in Y, and a factorisation

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[dr, "g"'] & U \arrow[d, "h"] \\ & Z. \end{tikzcd}$$

Proof. Let $\Gamma \subset Y \times Z$ be the image of (f, g). Then the projection morphism $p\colon \Gamma \longrightarrow Y$ is proper, and $p^{-1}(y) = (y, z)$, is a single point, by assumption. It follows that p is finite over an open neighbourhood U of $y \in Y$. But

$$f_*\mathcal{O}_{f^{-1}(U)} \supset p_*\mathcal{O}_{p^{-1}U} \supset \mathcal{O}_U = f_*\mathcal{O}_{f^{-1}(U)}.$$

It follows that $p|_{p^{-1}(U)}$ is an isomorphism. Let h be the composition of the inverse map with the projection down to Z. $\square$

Lemma 14.4. *Let $F\colon C\times B_0 \longrightarrow Y$ be a morphism such that $F(p,u) = y$, for all $b \in B_0$, where C is a smooth proper curve and B_0 is an open subset of a smooth proper curve B.*

If there is a point $q \in C$ such that $F|_{\{q\}\times B_0}$ is not constant then the rational map $F\colon C \times B \dashrightarrow Y$ is not defined at some point of $\{p\} \times B$. In particular Y contains a rational curve passing through x.

Proof. If F were a morphism on the whole of $C \times B$, then by the rigidity Lemma, we could find an open neighbourhood U of p and an open neighbourhood V of $\{p\} \times B$ such that $F(p_1, b_1)$ is independent of b_1, for all $(p_1, b_1) \in V$. But then $F(q, b)$ is independent of b, a contradiction. □

So it remains to show that we can bend f.

Theorem 14.5. *Let $f\colon C \longrightarrow X$ be a morphism of a smooth curve into a variety X, such that $f(C)$ is contained in the smooth locus.*

Then

$$T_f \operatorname{Hom}(C,X) = H^0(C, f^*T_X) \quad \text{and} \quad T_f \operatorname{Hom}(C,X,p,x) = H^0(C, f^*T_X \otimes \mathcal{I}_p).$$

Moreover,

$$\dim_f \operatorname{Hom}(C,X) \geq \chi(C, f^*T_X) \quad \text{and} \quad \dim_f \operatorname{Hom}(C,X,p,x) \geq \chi(C, f^*T_X \otimes \mathcal{I}_p).$$

Here the second space represents the set of morphisms which send p to x. The point here is that the spaces $\operatorname{Hom}(C,X)$ are naturally schemes not varieties. So even if the spaces were smooth at f, there might be obstructions to deforming f, beyond the first level (which are represented by the tangent space). But the obstructions to deforming f live in $H^1(C, f^*T_X)$. Thus the difference

$$\chi(C, f^*T_X) = h^0(C, f^*T_X) - h^1(C, f^*T_X),$$

represents deformations which can be lifted to any level, whence the inequality on dimensions.

We need to recall Hirzebruch-Riemman-Roch for curves:

Theorem 14.6. *Let E be a vector bundle of rank r and degree d over a smooth curve C.*

Then

$$\chi(C,E) = h^0(C,E) - h^1(C,E) = d - r(g-1).$$

In our case, $-K_X \underset{f}{\cdot} C$ is positive and the degree is either $-K_X \underset{f}{\cdot} C$ or $-K_X \underset{f}{\cdot} C - n$ (if one wants to fix a point). But there seems to no way to ensure that the degree is more than $n(g-1)$. In other words, why should the genus be small in relation to the degree? Note also, that we want to deform f in a non-trivial way (that is, not simply by applying an automorphism of C). In other words we want more deformations than the dimension the automorphism group (equal to $h^0(C, T_C)$).

If C is a copy of $\mathbb{P}^1$, one can compose f with the morphism $z \longrightarrow z^n$. This has the effect of multiplying the degree by n, without changing the genus.

If C is an elliptic curve, we can still play the same trick. Multiplication by n defines an isogeny of degree n^2. This has the same effect of increasing the degree by a factor of n^2, whilst leaving the genus unchanged.

But now suppose that the genus is at least 2. Let $\pi\colon D \longrightarrow C$ be a generically smooth morphism of degree e. Then

$$2h - 2 = e(2g - 2) + b.$$

The best one can hope for is that π is étale, that is, $b = 0$. The problem is that then the genus increases by the same factor as the degree (they both go up by a factor of e).

Suppose for a minute that the characteristic is p. Consider Frobenius $F\colon C \longrightarrow C$. Composing f with Frobenius has the effect of increasing the degree, without changing the genus. So, applying a high enough power of Frobenius, we may assume that the morphism f deforms, fixing the fact that p maps to x. But then X must be covered by rational curves.

Now the case when C is a rational curve is a little special. Since $\mathbb{P}^1$ has automorphisms, to break C, we need to deform it keeping two points fixed. This wastes a little more of the degree. In fact

$$\chi(\mathbb{P}^1, f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-p-q)) = -K_X \underset{f}{\cdot} C - 2n.$$

Playing around with this a little, one sees that one can break a rational curve on a Fano variety until its degree is no more than $n+1$.

To summarise. If the characterstic is not zero, then we can find rational curves C of degree $-K_X \cdot C \leq n+1$.

In the general case, we realise X and C as schemes over finitely generated extension K of $\mathbb{Q}$ (just embed X into projective space, and let K be the field generated any set of defining equations). Pick an integral domain R, with field of fractions K, a finitely generated extension of $\mathbb{Z}$ and realise X over $\operatorname{Spec} R$ (that is, clear denominators from the definining equations).

For every prime p, consider the reduction X_p of X modulo p (just reduce the equations modulo this prime). The residue field is a finitely generated extension of $\mathbb{F}_p$, whence a finite field. For all but finitely many primes, X_p is smooth along C_p and $-K_{X_p} \cdot C_p < 0$, and if X is a smooth Fano, then so is X_p.

Thus by what we have already proved, X_p is covered by rational curves of bounded degree, for all but finitely many primes. Since the Hilbert scheme of subvarieties of bounded degree is of finite type, the same must hold at the generic point, that is, in characteristic zero.

Note that it is absolutely crucial that the rational curves we produce are of bounded degree, else we could never return from characteristic p to characteristic zero. To prove (14.1) one needs to work a little harder. As X is not smooth and $-K_X$ is not ample, one cannot assume that the rational curves we produce are K_X-negative. To compensate, one composes with a larger power of Frobenius, but at the same time fixes as many points as possible. This way we break off many curves. Playing around with the Hodge Index Theorem, one gets the indicated bound, in characteristic p, at least when M is an ample $\mathbb{Q}$-divisor. One then lifts this result to all characteristics. The general case follows easily from the case when M is ample by an easy limiting argument.

Corollary 14.7. *Let X be a normal projective variety of dimension n. Let M be any nef divisor. Suppose that we may find nef $\mathbb{R}$-divisors $D_1, D_2, \dots, D_n$ with the following two properties:*

(1) $D_1 \cdot D_2 \cdot \dots \cdot D_n = 0$, and

(2) $-K_X \cdot D_2 \cdot \dots \cdot D_n > 0$.

Then X is swept out by rational curves Σ, such that $D_1 \cdot \Sigma = 0$ and

$$M \cdot \Sigma \leq 2n \frac{M \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}.$$

Proof. Let $H_1, H_2, \dots, H_n$ be ample $\mathbb{Q}$-divisors. If we pick H_2, H_3, $\dots H_n$ close enough to D_2, D_3, $\dots$, D_n, we have

$$-K_X \cdot H_2 \cdot H_3 \cdots \cdot H_n > 0.$$

Pick positive integers m_i such that $m_i H_i$ is very ample, $2 \leq i \leq n$ and let C be the intersection of general elements of $|m_i H_i|$. Then C is contained in the smooth locus of X and $-K_X \cdot C > 0$. By (14.1) there is a rational curve Σ such that

$$\begin{aligned}(kD + H) \cdot \Sigma &\leq 2n \frac{(kD + H) \cdot (m_2 H_2) \cdot (m_3 H_3) \cdot \dots \cdot (m_n H_n)}{-K_X \cdot (m_2 H_2) \cdot (m_3 H_3) \cdot \dots \cdot (m_n H_n)} \\ &= 2n \frac{(kD + H) \cdot H_2 \cdot H_3 \cdot \dots \cdot H_n}{-K_X \cdot H_2 \cdot H_3 \cdot \dots \cdot H_n}.\end{aligned}$$

where k is a positive integer. As H_2, H_3, $\dots H_n$ approach D_2, D_3, $\dots$, D_n, the numerator and denominator approach positive constants. Thus the left hand side is bounded, and as we vary k, $\Sigma = \Sigma_k$ belongs to a bounded family. Thus we may as well assume that Σ is fixed. Letting k go to infinity and H approach M gives the result. $\square$

Let me end this section by mentioning three fabulous results:

Theorem 14.8 (Mori)**.** *Let X be a smooth projective variety.*

If T_X is ample then $X \simeq \mathbb{P}^n$.

Theorem 14.9 (Cho-Miyaoka-Shepherd-Barron; Kebekus)**.** *Let X be a smooth projective Fano variety of dimension n.*

If the smallest degree of a covering family of rational curves is $n+1$ then $X \simeq \mathbb{P}^n$.

Note that (14.9) implies (14.8). Indeed, if C is a rational curve, then

$$T_X|_C = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \dots \mathcal{O}_{\mathbb{P}^1}(a_n),$$

by a result of Grothendieck (every vector bundle on $\mathbb{P}^1$ splits) and if $T_X|_C$ is ample then $a_i \geq 1$. But then

$$-K_X \cdot C = 2 + \sum a_i \geq n+1.$$

Theorem 14.10 (Bogomolov-McQuillan; Kebekus-Solá Conde-Toma)**.** *Let $\mathcal{F} \subset T_X$ be a possibly singular foliation on a normal projective variety.*

If C is any curve contained in the smooth locus of X along which $\mathcal{F}$ is a regular foliation, then the leaves of $\mathcal{F}$ through any point $p \in C$ are algebraic and if $x \in C$ is general, or $\mathcal{F}$ is regular, then this leaf is rationally connected.