17. The MMP

Using the cone (16.6) and contraction theorem (16.3), one can define the MMP in all dimensions:

(1) Start with a kawamata log terminal pair (X, Δ).
(2) Is $K_X + \Delta$ nef? Is yes, then stop.
(3) Otherwise by (16.6) there is an extremal ray R of the cone of curves $\overline{\mathrm{NE}}(X)$ on which $K_X + \Delta$ is negative. By (16.3) there is a contraction $\pi\colon X \longrightarrow Z$ of R.

Mori fibre space: If $\dim Z \leq \dim X$ then the fibres of π are Fano varieties.

Birational contraction: There are two cases:

Divisorial: π contracts a divisor. Replace X by Z and return to (2).

Small: π is small. In this case we cannot replace X by Z.

Lemma 17.1. *Let $\pi\colon X \longrightarrow Z$ be a proper small contraction and let D be $\mathbb{R}$-Cartier.*

If $\pi_* D$ is $\mathbb{R}$-Cartier then $D \cdot C = 0$ for all curves C contracted by π.

Proof. If $E = \pi_* D$ is $\mathbb{R}$-Cartier, then we can pull it back, $\pi^* E = \pi^* \pi_* D$. Now outside of the exceptional locus, D and $\pi^* \pi_* D$ are equal. But then they must be equal, since the exceptional locus is of codimension two or more. Thus

$$D = \pi^* \pi_* D.$$

In particular $D \cdot C = 0$ for all curves C contracted by π. □

Definition 17.2. *Let $\pi\colon X \longrightarrow Z$ be a small extremal contraction, such that $-(K_X + \Delta)$ is π-ample and X is $\mathbb{Q}$-factorial. The **flip** of π, $\pi^+\colon X^+ \longrightarrow Z$ is a small extremal contraction such that $K_{X^+} + \Delta^+$ is π^+-ample and X^+ is $\mathbb{Q}$-factorial.*

Theorem 17.3 (Existence)**.** *Flips exist.*

Using (17.3) we simply replace X by X^+. This raises another issue:

Conjecture 17.4 (Termination)**.** *There is no infinite sequence of flips.*

The problem is that there is no obvious topological reason why we cannot have an infinite sequence of flips.

However if we aim for less than we do get termination. We define the MMP with scaling:

(1) Start with a kawamata log terminal pair (X, Δ), where C is π-big and $K_X + \Delta + C$ is π-nef.

(2) Let $\lambda \geq 0$ be the nef threshold, so that $K_X + \Delta + \lambda C$ is π-nef. Is $\lambda = 0$? Is yes, then stop. $K_X + \Delta$ is nef.
(3) Otherwise by (16.6) there is an extremal ray R of the cone of curves $\overline{\mathrm{NE}}(X)$ on which $K_X + \Delta$ is negative and $K_X + \Delta + C$ is zero. By (16.3) there is a contraction $\pi\colon X \longrightarrow Z$ of R.
   Mori fibre space: If $\dim Z \leq \dim X$ then the fibres of π are Fano varieties.
   Birational contraction: There are two cases:
      Divisorial: π contracts a divisor. Replace X by Z and return to (2).
      Flip: π is small. In this case we replace X by X^+ the flip.

Note that the MMP with scaling is somewhat similar to the relative MMP; using the divisor C we narrow down the set of extremal rays we pick as we run the MMP.

Theorem 17.5 (Termination)**.** *The MMP with scaling always terminates.*

Note that there is an implied big condition which is crucial to the proof. We end with two applications of the MMP with scaling. In the appendix to Hartshorne there is an example due to Hironaka of a smooth proper threefold which is not projective. It is rational, birational to $\mathbb{P}^3$.

Definition 17.6. *Let M be a smooth compact complex manifold.*

We say that M is ***Moishezon*** *if the transcendence degree of the space of meromorphic functions is equal to the dimension.*

In fact M is Moishezon if and only if it is birational to a projective variety. More generally one can extend the notion of Moishezon manifolds to analytic spaces. By a result due to Artin these are the same as algebraic spaces.

Theorem 17.7. *Let $f\colon X \longrightarrow Z$ be a proper morphism of algebraic spaces (or Moishezon spaces).*

If $K_X + \Delta$ is kawamata log terminal and X is $\mathbb{Q}$-factorial, and f does not contract any rational curves then f is a minimal model. In particular f is projective.

Proof. Let $\pi\colon Y \longrightarrow X$ be a log resolution. We may write

$$K_Y + \Gamma' = \pi^*(K_X + \Delta) + E,$$

where $\Gamma' \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma' = \Delta$ and $\pi_*E = 0$. Then $\llcorner \Gamma' \lrcorner = 0$. Pick $F \geq 0$ supported on the exceptional locus such that $\llcorner \Gamma' + F \lrcorner = 0$. Then $(Y, \Gamma = \Gamma' + F)$ is kawamata log terminal, and

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E + F.$$

Pick H ample such that $K_Y + \Gamma + H$ is nef and kawamata log terminal. We run the $(K_Y + \Gamma)$-MMP with scaling over Z.

Suppose that some step of this MMP is not over X. Then we get $g \colon Y \longrightarrow Y'$ and $Y' \dashrightarrow X$ is not a morphism. But then X contains a rational curve, the image of a rational curve contracted by f, a contradiction. So the MMP over Z is automatically a MMP over X. As π is birational, the MMP with scaling over X always terminates (every divisor is big over X).

At the end, negativity of contraction implies that $Y \longrightarrow X$ is the identity. But then f is a log terminal model. □

Definition 17.8. *We say that G is* ***Jordan*** *if there is a an integer m such that if $H \subset G$ is any finite subgroup then there is a normal abelian subgroup $A \lhd H$ of index at most m.*

Note that it suffices to exhibit an abelian subgroup $A \subset H$ of bounded index. The kernel of the natural action of H on the left cosets of A in H is an abelian normal subgroup of index at most $m!$.

Theorem 17.9 (Jordan)**.** $\mathrm{GL}_n(\mathbb{C})$ *is Jordan.*

Corollary 17.10. $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{C})$ *is Jordan.*

Proof. $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a subgroup of $\mathrm{GL}_m(\mathbb{C})$, some m. □

Conjecture 17.11 (Serre)**.** $\mathrm{Gal}(K(x_0, x_1, \dots, x_n)/K) = \mathrm{Bir}(\mathbb{P}^n)$ *is Jordan.*

Conjecture 17.12 (Borisov-Alexeev-Borisov)**.** *Fix n and $\epsilon > 0$.*

The family of all projective varieties of dimension n such that $-K_X$ is ample and the log discrepancy is at least ϵ is bounded.

Example 17.13. *Let S be the cone over a rational normal curve of degree n.*

Then S is a projective surface with quotient singularities. Varying n we certainly don't get a bounded family. Let $\pi \colon T \longrightarrow S$ be the minimal resolution. Then $T \simeq \mathbb{F}_n$, the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, with a section E of self-intersection $-n$. T has Picard number two so that S has Picard number one. If we write

$$K_T + E = \pi^* K_S + aE,$$

then $a = 2/n$ so that

$$-K_T - \frac{n-2}{n}E = \pi^*(-K_S).$$

Let L be the image of a fibre F. We have

$$\begin{aligned}-K_S \cdot L &= -\pi^* K_S \cdot F \\ &= -K_T \cdot F - \frac{n-2}{n} E \cdot F \\ &= 2 - \frac{n-2}{n} > 0.\end{aligned}$$

As the Picard rank of S is 1 it follows that $-K_S$ is ample. Of course the log discrepancy is going to zero.

We have the following result of Prokhorov and Shramov:

Theorem 17.14. $(17.12)_n$ *implies* $(17.11)_n$.

We sketch the proof. We will in fact prove much more. We will prove the same result provided X is rationally connected and in this case we will exhibit a universal m, depending only on the dimension. As part of the induction we will prove that there is a point fixed by a subgroup of bounded index.

Lemma 17.15. *Let X be a quasi-projective variety and let G be a finite subgroup of the birational automorphism group of X.*

Then we may find a smooth projective variety Y birational to X such that $G \subset \operatorname{Aut}(Y)$.

Proof. We may assume that X is projective. Let $L = K(X)$ be the function field of X and let $M = L^G$ be the fixed field. Then $Z = X/G$ is a variety with functional field M. Let Y be the normalisation of Z in L. □

According to (17.15) we may assume that if G is a finite group then it is a subgroup of the automorphism group. We run the G-equivariant K_X-MMP, with scaling of an ample divisor H. Since X is rationally connected and X is smooth we will never get to a minimal model. Therefore we never get to the case $t = 0$. Every step of the K_X-MMP with scaling of H is also a step of the $(K_X + \Delta)$-MMP with scaling of H, where $\epsilon > 0$ is sufficiently small, so that $(X, \Delta = \epsilon H)$ is kawamata log terminal. Thus this MMP always terminates.

At the end we have a Mori fibre space, $\pi\colon X \longrightarrow Z$. G acts on both X and Z and π is equivariant. There are two cases. If $\dim Z > 0$ then let F be the generic fibre. There is an exact sequence

$$0 \longrightarrow G_0 \longrightarrow G \longrightarrow G_1 \longrightarrow 0$$

where $G_0 \subset \mathrm{Aut}(F)$ and $G_1 \subset \mathrm{Aut}(Z)$. F is a Fano variety and so it is rationally connected. Z is rationally connected as it is the image of X. It is not hard to check we are done by induction on the dimension.

Otherwise Z is a point and X is a Fano variety of Picard number one. As we ran the K_X-MMP, X has terminal singularities. By $(17.12)_n$, applied with $\epsilon = 1$, X belongs to a bounded family. It follows that there is some fixed m and N such that $-mK_X$ is very ample and embeds X in $\mathbb{P}^N$. In this case $G \subset \mathrm{PGL}_{N+1}(\mathbb{C})$.

The major unsolved conjecture would seem to be:

Conjecture 17.16 (Abundance)**.** *Let* (X, Δ) *be a kawamata log terminal pair.*

Then $K_X + \Delta$ *is nef if and only if it is semiample.*

Challenge case: Prove (17.16) when $\Delta = 0$ and X is a smooth surface S, $q = 0$ and $\nu = 1$, without cheating (e.g using the fact that $\chi(S, \mathcal{O}_S) > 0$). The key point is to exhibit elliptic curves through every point.

The main point is to prove:

Conjecture 17.17. *Let* X *be a smooth projective variety.*

Then either

(1) $\kappa(X) \geq 0$, *or*

(2) X *is uniruled.*