3. Ample and Semiample

We recall some very classical algebraic geometry. Let D be an integral Weil divisor. Provided $h^0(X, \mathcal{O}_X(D)) > 0$, D defines a rational map:

$$\phi = \phi_D \colon X \dashrightarrow Y.$$

The simplest way to define this map is as follows. Pick a basis $\sigma_0, \sigma_1, \dots, \sigma_m$ of the vector space $H^0(X, \mathcal{O}_X(D))$. Define a map

$$\phi \colon X \longrightarrow \mathbb{P}^m \qquad \text{by the rule} \qquad x \longrightarrow [\sigma_0(x) : \sigma_1(x) : \cdots : \sigma_m(x)].$$

Note that to make sense of this notation one has to be a little careful. Really the sections don't take values in $\mathbb{C}$, they take values in the fibre L_x of the line bundle L associated to $\mathcal{O}_X(D)$, which is a 1-dimensional vector space (let us assume for simplicity that D is Cartier so that $\mathcal{O}_X(D)$ is locally free). One can however make local sense of this morphism by taking a local trivialisation of the line bundle $L|_U \simeq U \times \mathbb{C}$. Now on a different trivialisation one would get different values. But the two trivialisations differ by a scalar multiple and hence give the same point in $\mathbb{P}^m$.

However a much better way to proceed is as follows.

$$\mathbb{P}^m \simeq \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*).$$

Given a point $x \in X$, let

$$H_x = \{\, \sigma \in H^0(X, \mathcal{O}_X(D)) \,|\, \sigma(x) = 0 \,\}.$$

Then H_x is a hyperplane in $H^0(X, \mathcal{O}_X(D))$, whence a point of

$$\phi(x) = [H_x] \in \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*).$$

Note that ϕ is not defined everywhere. The problem in either description is that the sections of $H^0(X, \mathcal{O}_X(D))$ might all vanish at x.

Definition 3.1. *Let X be a normal variety and let D be an integral Weil divisor. The* ***complete linear system***

$$|D| = \{\, D' \,|\, D' \geq 0, D' \sim D \,\}.$$

The ***base locus*** *of $|D|$ is the intersection of all of the elements of $|D|$. We say that $|D|$ is* ***base point free*** *if the base locus is empty.*

In fact, given a section $\sigma \in H^0(X, \mathcal{O}_X(D))$ the zero locus D' of σ defines an element of $|D|$ and every element of $|D|$ arises in this fashion. Thus the base locus of $|D|$ is precisely the locus where ϕ_D is not defined. Thus $|D|$ is base point free if and only if ϕ_D is a morphism (or better everywhere defined). In this case the linear system can be recovered by pulling back the hyperplane sections of $Y \subset \mathbb{P}^{m-1}$ and in fact $\mathcal{O}_X(D) = \phi^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Definition 3.2. *We say that a $\mathbb{Q}$-divisor D on a normal variety is* ***semiample*** *if $|mD|$ is base point free for some $m \in \mathbb{N}$. We say that an integral divisor D is* ***very ample*** *if $\phi\colon X \longrightarrow \mathbb{P}^n$ defines an embedding of X. We say that D is* ***ample*** *if mD is very ample for some $m \in \mathbb{N}$.*

Theorem 3.3 (Serre-Vanishing)**.** *Let X be a normal projective variety and let D be a Cartier divisor on X.*

TFAE

(1) *D is ample.*
(2) *For every coherent sheaf $\mathcal{F}$ on X, there is a positive integer m such that*
$$H^i(X, \mathcal{F}(mD)) = 0,$$
for all $m \geq m_0$ and $i > 0$ (and these cohomology groups are finite dimensional vector spaces).
(3) *For every coherent sheaf $\mathcal{F}$ on X, there is a positive integer m_0 such that the natural map*
$$H^0(X, \mathcal{F}(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{F}(mD),$$
is surjective, for all m divisible by m_0.

Proof. Suppose that D is ample. Then $\phi_{kD} = i\colon X \longrightarrow \mathbb{P}^n$ defines an embedding of X into projective space, for some $k \in \mathbb{N}$. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{F}_j = \mathcal{F}(jD)$, $0 \leq j \leq k-1$. Let $m \in \mathbb{N}$. Then we may write $m = m'k + j$, for some $0 \leq j \leq k-1$. In this case
$$H^i(X, \mathcal{F}(mD)) = H^i(X, \mathcal{F}_j(m'kD)) = 0,$$
Thus replacing D by kD we may assume that $D = i^*H$ is very ample. In this case
$$H^i(X, \mathcal{F}(mD)) = H^i(\mathbb{P}^n, i_*\mathcal{F}(mH)).$$
Thus replacing X by $\mathbb{P}^n$, $\mathcal{F}$ by $i_*\mathcal{F}$ and D by H we may assume that $X = \mathbb{P}^n$.

We proceed by descending induction on i. If $i > n$ the result is clear (either use Grothendieck's theorem or the fact that $\mathbb{P}^n$ is covered by $n+1$ open affines).

Now every coherent sheaf on $\mathbb{P}^n$ is a quotient of a direct sum of line bundles. Thus we get a short exact sequence:
$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$
where $\mathcal{E}$ is a direct sum of line bundles and $\mathcal{K}$ is coherent. Since we have shown that $A_{n-1}(\mathbb{P}^n) = \mathbb{Z}$ it follows that each line bundle on $\mathbb{P}^n$ is of the form $\mathcal{O}_{\mathbb{P}^n}(aH)$ for some $a \in \mathbb{Z}$ (it is customary to drop the

H). Twisting by a large multiple of H we may assume that each $a > 0$ is large. By direct computation

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) = 0,$$

for $i > 0$. Taking the long exact sequence associated to the short exact sequence gives

$$H^i(\mathbb{P}^n, \mathcal{F}) \simeq H^{i+1}(\mathbb{P}^n, \mathcal{K}),$$

and so (1) implies (2) by descending induction on i.

Suppose that (2) holds. Let $p \in X$ be a point of X and let $\mathbb{C}_p$ be the skyscraper sheaf supported at p, with stalk $\mathbb{C}$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I}(mD) \longrightarrow \mathcal{F}(mD) \longrightarrow \mathcal{F}(mD) \otimes \mathbb{C}_p \longrightarrow 0,$$

where $\mathcal{I}(mD)$ is a coherent sheaf. Since by assumption

$$H^1(X, \mathcal{I}(mD)) = 0,$$

for m sufficiently large, taking the long exact sequence of cohomology, it follows that

$$H^0(X, \mathcal{F}(mD)) \longrightarrow H^0(X, \mathcal{F}(mD) \otimes \mathbb{C}_p),$$

is surjective for m sufficiently large. But then

$$H^0(X, \mathcal{F}(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{F}(mD),$$

is surjective in a neighbourhood of p by Nakayama's Lemma. As X is compact, it follows that (2) implies (3).

Suppose that (3) holds. Pick m_0 such that

$$H^0(X, \mathcal{O}_X(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(mD),$$

is surjective for all $m \geq m_0$. Since there is a surjection

$$\mathcal{O}_X(mD) \longrightarrow \mathbb{C}_p(mD),$$

it follows that there is a surjection,

$$H^0(X, \mathcal{O}_X(mD)) \otimes \mathcal{O}_X \longrightarrow \mathbb{C}_p(mD).$$

But then we may find a section $\sigma \in H^0(X, \mathcal{O}_X(mD))$ not vanishing at x. This gives an element $D' \in |mD|$ not containing x. Thus $x \notin B_m$. It follows that B_m is empty, so that mD is base point free and so D is semiample.

Let $x \neq y$ be two points of X and let $\mathcal{I}_x$ be the ideal sheaf of x. Pick a positive integer m_0 such that

$$H^0(X, \mathcal{I}_x(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_x(mD),$$

is surjective for all $m \geq m_0$. As above, it follows that

$$H^0(X, \mathcal{I}_x(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_x \otimes \mathbb{C}_y(mD) = \mathbb{C}_y(mD),$$

is surjective. But then we may an element of $|mD|$ passing through x but not through y. Thus $\phi_{mD}(x) \neq \phi_{mD}(y)$. The same m_0 works for all $y \neq x$. Given x and y we may find a neighbourhood of $(x, y) \in X \times X$ such that ph_{mD} is injective on this neighbourhood. By compactness of $X \times X$ we may find m_0 that works for all $x \neq y$. It follows that ϕ_{mD} is injective, that is $|mD|$ separates points.

Finally pick a length two scheme $z \subset X$ supported at x. This determines a surjection,

$$\mathcal{I}_x(mD) \longrightarrow \mathbb{C}_x(mD)$$

Composing, we get a surjection

$$H^0(X, \mathcal{I}_x(mD)) \otimes \mathcal{O}_X \longrightarrow \mathbb{C}_x(mD),$$

By a repeat of the Noetherian argument given above, if m is sufficiently large and divisible then this map is surjective for all irreducible length two schemes. On the other hand, the Zariski tangent space of any scheme is the union of the irreducible length two subschemes of X, and so the map ϕ_{mD} is an injective immersion (that is it is injective on tangent spaces). But then ϕ_{mD} is an embedding (it is a straightforward application of Nakayama's Lemma to check that the implicit function argument extends to the case of singular varieties). Thus (3) implies (1). □

More generally if X is a scheme, we will use (2) of (3.3) as the definition of ample:

Lemma 3.4. *Let $f \colon X \longrightarrow Y$ be a morphism of projective schemes. Let D be a $\mathbb{Q}$-Cartier divisor on Y.*

(1) If D is ample and f is finite then f^*D is ample.
(2) If f is surjective and f^*D is ample (this can only happen if f is finite) then D is ample.

Proof. We may suppose that D is Cartier.

Suppose that D is ample and let $\mathcal{F}$ be a coherent sheaf on X. As f is finite, we have

$$H^i(X, \mathcal{F}(mf^*D)) = H^i(Y, f_*\mathcal{F}(mD)),$$

which is zero for m sufficiently large. Thus f^*D is ample. This proves (1).

Now suppose that f is surjective and f^*D is ample. We first prove (2) in the special case when $X = Y_{\text{red}}$ and f is the natural inclusion. Suppose that f^*D is ample. Let $\mathcal{I}_X$ be the ideal sheaf of X in Y. Then $\mathcal{I}_X^k = 0$ for some $k > 0$, which we may assume to be minimal. We proceed by induction on k. If $k = 1$ then f is an isomorphism and there is nothing to prove. Otherwise $k > 1$. Let $\mathcal{F}$ be a coherent sheaf on Y. Then there is an exact sequence

$$0 \longrightarrow \mathcal{H}(mD) \longrightarrow \mathcal{F}(mD) \longrightarrow \mathcal{G}(mD) \longrightarrow 0,$$

where $\mathcal{G} = \mathcal{F} \otimes \mathcal{O}_X$ and $\mathcal{H}(mD)$ is coherent. Then for m sufficient large,

$$h^i(Y, \mathcal{F}(mD)) = h^i(Y, \mathcal{H}(mD)).$$

On the other hand, $\mathcal{H}$ is a coherent sheaf supported on the proper subscheme $Y' \subset Y$ defined by $\mathcal{I}^{k-1}$, and we are done by induction on k.

Let $X' = X_{\text{red}}$ and $Y' = Y_{\text{red}}$. Since taking the reduction is a functor, there is a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y, \end{array}$$

where $f' = f_{\text{red}}$. We may pullback D to X' either way around the square and we get the same answer both ways. It follows that to prove that D is ample, it suffices to prove that $D' = D_{\text{red}}$ is ample. Thus, replacing $f\colon X \longrightarrow Y$ by $f'\colon X' \longrightarrow Y'$, we may assume that X and Y are projective varieties.

Now suppose that $Y = Y_1 \cup Y_2$ is the union of two closed subvarieties, let X be their disjoint union and let $f\colon X \longrightarrow Y$ be the obvious morphism. If $\mathcal{F}$ is any coherent sheaf, then there is an exact sequence

$$0 \longrightarrow \mathcal{F}(mD) \longrightarrow f^*\mathcal{F}(mD) = \mathcal{F}_1(mD) \oplus \mathcal{F}_2(mD) \longrightarrow \mathcal{G}(mD) \longrightarrow 0,$$

where $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{O}_{Y_i} = \mathcal{F}_{Y_i}$ is the restriction of $\mathcal{F}$ to Y_i and $\mathcal{G}$ is a sheaf supported on $Z = Y_1 \cap Y_2$. By (1) applied to either of the inclusions $Z \longrightarrow Y_i$, $D|_Z$ is ample. Moreover for m sufficiently large both of the natural maps,

$$H^0(Y_i, \mathcal{F}_i(mD)) \longrightarrow H^0(Z, \mathcal{G}(mD)),$$

are surjective. Taking the long exact sequence of cohomology, we see that

$$H^i(X, \mathcal{F}(mD)) = 0,$$

is zero, for $i > 0$ and m sufficiently large, since it is easy to see directly that

$$H^0(Y_1, \mathcal{F}_1(mD)) \oplus H^0(Y_2, \mathcal{F}_2(mD)) \longrightarrow H^0(Z, \mathcal{G}(mD)),$$

is surjective, for m sufficiently large.

Using the same argument and the same square as before (but where X' and Y' are now the irreducible components of X and Y) we may assume that X and Y are integral. Now the proof proceeds by induction on the dimension of X (and so of Y).

Let $\mathcal{F}$ be a sheaf on Y.

Claim 3.5. *There is a sheaf $\mathcal{G}$ on X and a map*

$$f_*\mathcal{G} \longrightarrow \bigoplus_{i=1}^{k} \mathcal{F},$$

which is an isomorphism over an open subset of Y, where k is the degree of the map f.

Proof of (3.5). Note that k is the degree of the field extension $K(X)/K(Y)$. Pick an affine subset U of X and pick sections $s_1, s_2, \dots, s_k \in \mathcal{O}_U$ which generate the field extension $K(X)/K(Y)$. Let M be the coherent sheaf generated by the sections $s_1, s_2, \dots, s_k \in K(Y)$. Then there is a morphism of sheaves

$$\bigoplus_{i=1}^{k} \mathcal{O}_Y \longrightarrow f_*\mathcal{M},$$

which is an isomorphism over an open subset of Y. Taking $\mathrm{Hom}_{\mathcal{O}_Y}$, we get a morphism of sheaves

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{M}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\bigoplus_{i=1}^{k} \mathcal{O}_Y, \mathcal{F}) \simeq \bigoplus_{i=1}^{k} \mathcal{F}.$$

Finally observe that

$$\mathrm{Hom}(f_*\mathcal{M}, \mathcal{O}_Y) = f_*\mathcal{G},$$

for some coherent sheaf $\mathcal{G}$ on X. □

Thus there is an exact sequence

$$0 \longrightarrow \mathcal{K}(mD) \longrightarrow f_*\mathcal{G}(mD) \longrightarrow \bigoplus_{i=1}^{k} \mathcal{F}(mD) \longrightarrow \mathcal{Q}(mD) \longrightarrow 0,$$

where $\mathcal{K}$ and $\mathcal{Q}$ are defined to preserve exactness. We may break this exact sequence into two parts,

$$0 \longrightarrow \mathcal{H}(mD) \longrightarrow \bigoplus_{i=1}^{k} \mathcal{F}(mD) \longrightarrow \mathcal{Q}(mD) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{K}(mD) \longrightarrow f_*\mathcal{G}(mD) \longrightarrow \mathcal{H}(mD) \longrightarrow 0,$$

where $\mathcal{H}$ is a coherent sheaf. Note that the support of the sheaves $\mathcal{K}$ and $\mathcal{Q}$ is smaller than the support of $\mathcal{F}$. But then by induction any twist of their higher cohomology must vanish, for m sufficiently large. Since the higher cohomology of $f_*\mathcal{G}(mD)$ also vanishes, as f^*D is ample, looking at the first exact sequence, we have that

$$H^i(Y, \mathcal{H}(mD)) = 0,$$

for $i > 0$ and m sufficiently large. Looking at the second exact sequence, it follows that

$$H^i(X, \mathcal{F}(mD)) = 0,$$

for all m sufficiently large. □

Lemma 3.6. *Let X be a projective variety and let D be a $\mathbb{Q}$-Cartier divisor.*

If H is an ample divisor then there is an integer m_0 such that $D+mH$ is ample for all $m \geq m_0$.

Proof. By induction on the dimension n of X. By (3.4) we may assume that X is normal. It is straightforard to check that a divisor on a curve is ample if and only if its degree is positive. In particular this result is easy for $n \leq 1$.

So suppose that $n > 1$. Replacing D and H by a multiple we may assume that D is Cartier, so that it is in particular an integral Weil divisor and we may assume that H is very ample. Let $x \neq y \in X$ be any two points of X and let $Y \in |H|$ be a general element containing x and y. Then Y is a projective variety and there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(kD+(m-1)H) \longrightarrow \mathcal{O}_X(kD+mH) \longrightarrow \mathcal{O}_Y(kD+mH) \longrightarrow 0.$$

By induction on the dimension there is an integer m_0 such that $(D + mH))|_Y$ is ample for all $m \geq m_0$. Pick a positive integer k such that $k(D+mH)|_Y$ is very ample. By Serre vanishing, possibly replacing m_0 by a larger integer, we may assume that $(kD + mH)|_Y$ is very ample and that

$$H^i(X, \mathcal{O}_X(kD + mH)) = 0,$$

for all $m \geq m_0$ and $i > 0$. In particular

$$H^0(X, \mathcal{O}_X(kD + mH)) \longrightarrow H^0(Y, \mathcal{O}_Y(kD + mH)),$$

is surjective. By assumption we may find $B \in |(kD + mH)|_Y|$ containing x but not containing y and we may lift this to $B' \in |(kD + mH)|$ containing x by not containing y. It follows that $\phi_{kD+mH}\colon X \longrightarrow \mathbb{P}^N$ is an injective morphism, whence it is a finite morphism. But then $kD + mH$ is the pullback of an ample divisor under a finite morphism and so it must be ample. □

Definition-Lemma 3.7. *Let X be a variety and let $L/K(X)$ be a finite field extension.*

We may find a normal variety Y and a finite morphism $f\colon Y \longrightarrow X$ such that $K(Y)/K(X)$ is isomorphic to $L/K(X)$.

Proof. We may cover X finitely many open affine varieties $U_i = \operatorname{Spec} A_i$. Let B_i be the integral closure of A_i in L/K. Then $V_i = \operatorname{Spec} B_i$ is an affine variety there is a finite morphism $V_i \longrightarrow U_i$ and $K(V_i)/K(U_i) \simeq L/K(X)$. The variety Y is obtained by gluing V_i together. □

Definition-Theorem 3.8 (Stein factorisation)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of normal varieties. There is a factorisation of $f = h \circ g$ into a morphism $g\colon X \longrightarrow Z$ with connected fibres and a finite morphism $h\colon Z \longrightarrow Y$.*

Proof. Let $Z = \operatorname{Spec}_Y f_*\mathcal{O}_X$. Then $h\colon Z \longrightarrow Y$ is finite and there is a morphism $g\colon X \longrightarrow Z$. By construction $g_*\mathcal{O}_X = \mathcal{O}_Z$. It follows by a result of Zariski that the fibres of g are connected. □

Lemma 3.9. *Let $f\colon X \longrightarrow Y$ be a morphism of projective varieties and let D be a $\mathbb{Q}$-Cartier divisor on Y.*

(1) If D is semiample then f^*D is semiample.
(2) If Y is normal, f is surjective and f^*D is semiample then D is semiample.

Proof. Suppose that D is semiample. Then $|mD|$ is base point free for some $m \in \mathbb{N}$. Pick $x \in X$. By assumption we may find $D' \in |mD|$ not containing $y = f(x)$. But then $f^*D' \in |mf^*D|$ does not contain x. Thus $|mf^*D|$ is base point free and so f^*D is semiample. This proves (1).

Now suppose that Y is normal, f is surjective and f^*D is semiample. Considering the Stein factorisation of f, we only need to prove (2) in the two special cases when f is finite and when f has connected fibres.

First assume that f is a finite morphism of normal varieties. Then we may find a morphism $g\colon W \longrightarrow X$ such that the composition $h = f \circ g\colon W \longrightarrow Y$ is Galois (take W to be the normalisation of X in the Galois closure of $K(X)/K(Y)$). By what we already proved g^*D is semiample. Replacing f by h, we may assume that f is Galois, so that $Y = X/G$ for some finite group G. Pick m so that mf^*D is base point free. Let $y \in Y$ and $x_1, x_2, \dots, x_k \in X$ be the finitely many points lying over y. Pick $\sigma \in H^0(X, \mathcal{O}_X(mf^*D))$ not vanishing at any one of the points $x_1, x_2, \dots, x_k$. Let $\sigma_1, \sigma_2, \dots, \sigma_l$ be the Galois conjugates of σ. Then each σ_i does not vanish at each x_j and $\prod_{i=1}^{l} \sigma_i$ is G-invariant, so that it is the pullback of a section $\tau \in H^0(Y, \mathcal{O}_Y(mlD))$ not vanishing at y. But then D is semiample.

Now suppose that the fibres of f are connected. Suppose that f^*D is semiample. Then $|mf^*D|$ is base point free for some $m > 0$. In this case

$$H^0(X, \mathcal{O}_X(mf^*D)) = H^0(Y, \mathcal{O}_Y(mD)).$$

Pick $y \in Y$. Let x be a point of the fibre $X_y = f^{-1}(y)$. Then there is a divisor $D' \in |mf^*D|$ not containing x. But $D' = f^*D''$ where $D'' \in |mD|$. Since D' does not contain x, D'' does not contain y. But then D is semiample. □