6. Ampleness criteria

We return to the problem of determining when a line bundle is ample.

**Theorem 6.1** (Nakai-Moishezon). Let $X$ be a normal projective variety and let $D$ be a $\mathbb{Q}$-Cartier divisor.

TFAE

1. $D$ is ample.
2. For every subvariety $V \subset X$ of dimension $k$,
   \[ D^k \cdot V > 0. \]

**Proof.** Suppose that $D$ is ample. Then $mD$ is very ample for some $m > 0$. Let $\phi: X \rightarrow \mathbb{P}^N$ be the corresponding embedding. Then $mD = \phi^*H$, where $H$ is a hyperplane in $\mathbb{P}^N$. Then
   \[ D^k \cdot V = \frac{1}{m^k}H^k \cdot \phi(V) > 0, \]
   since intersecting $\phi(V)$ with $H^k$ corresponds to intersecting $V$ with a linear space of dimension $N - k$. But this is nothing more than the degree of $\phi(V)$ in projective space.

Now suppose that $D$ satisfies (2). Let $H$ be a general element of a very ample linear system. Then we have an exact sequence
   \[ 0 \rightarrow \mathcal{O}_X(pD + (q - 1)H) \rightarrow \mathcal{O}_X(pD + qH) \rightarrow \mathcal{O}_H(pD + qH) \rightarrow 0. \]
By induction, $D|_H$ is ample. It is straightforward to prove that
   \[ h^i(H, \mathcal{O}_H(pD + qH)) = 0, \]
for $i > 0$, $p$ sufficiently large and any $q > 0$, by induction on the dimension. In particular,
   \[ h^i(X, \mathcal{O}_X(pD + (q - 1)H)) = h^i(X, \mathcal{O}_X(pD + qH)), \]
for $i > 1$, $p$ sufficiently large and any $q \geq 1$. By Serre vanishing the last group vanishes for $q$ sufficiently large. Thus by descending induction
   \[ h^i(X, \mathcal{O}_X(pD + qH)) = 0, \]
for all $q \geq 0$. Thus by (4.1) it follows that
   \[ h^0(X, \mathcal{O}_X(mD)) \neq 0, \]
for $m$ sufficiently large, that is, $|mD|$ is non-empty. As usual, this means that we may assume that $D \geq 0$ is Cartier. Let $\nu: \tilde{D} \rightarrow D_{\text{red}}$ be the normalisation of $D_{\text{red}}$, the reduced subscheme associated to $D$. Then $\nu^*D|_{D_{\text{red}}}$ is ample by induction. It follows by (3.4) that $D|_{D}$ is ample.
I claim that the map
\[ \rho_m : H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(D, \mathcal{O}_D(mD)), \]
is surjective for \( m \) sufficiently large. Consider the exact sequence
\[ 0 \longrightarrow \mathcal{O}_X((m - 1)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0. \]
As \( D|_D \) is ample,
\[ h^i(D, \mathcal{O}_D(mD)) = 0, \]
for \( i > 0 \) and \( m \) sufficiently large, by Serre vanishing. Thus
\[ h^1(X, \mathcal{O}_X(mD)) \leq h^1(X, \mathcal{O}_X((m-1)D)) \quad \text{and} \quad h^i(X, \mathcal{O}_X(mD)) = 0, \]
for \( i > 1 \), with equality if and only if \( \rho_m \) is surjective. Since
\[ h^1(X, \mathcal{O}_X(mD)), \]
is finite dimensional, its dimension cannot drop infinitely often, and so \( \rho_m \) is surjective as claimed.

As \( D|_D \) is ample, \( (mD)|_D \) is very ample. As we can lift sections, it follows that \( |mD| \) is base point free, that is, \( D \) is semiample. Let \( \phi = \phi_{mD} : X \longrightarrow \mathbb{P}^N \) be the corresponding morphism. Then \( D = \phi^*H \).

Suppose that \( C \) is a curve contracted by \( \phi \). Then
\[ D \cdot C = \phi^*H \cdot C = H \cdot \phi_*C = 0, \]
a contradiction. But then \( \phi_{mD} \) is a finite morphism and \( D = \phi^*H \) is ample by (3.4). \( \square \)

**Definition 6.2.** Let \( X \) be a normal projective variety and let \( D \) be a \( \mathbb{Q} \)-Cartier divisor. We say that \( D \) is **nef** if \( D \cdot C \geq 0 \) for all curves \( C \subset X \).

**Lemma 6.3.** Let \( X \) be a normal variety and let \( D \) be a \( \mathbb{Q} \)-Cartier divisor.

If \( D \) is semiample then \( D \) is nef.

**Proof.** By assumption there is a morphism \( \phi : X \longrightarrow Y \subset \mathbb{P}^n \) such that
\[ mD = \phi^*H. \]
But then
\[ D \cdot C = \frac{1}{m} \phi^*C \cdot H \geq 0. \]
\( \square \)

**Lemma 6.4.** Let \( X \) be a normal projective variety and let \( D \) be a \( \mathbb{Q} \)-Cartier divisor.

TFAE

1. \( D + H \) is ample for any ample \( \mathbb{Q} \)-divisor \( H \).
(2) If $V \subset X$ is any subvariety of $X$ then

$$D^k \cdot V \geq 0,$$

where $V$ has dimension $k$.

(3) $D$ is nef.

Proof. (1) implies (2) and (2) implies (3) are clear. (2) implies (1) follows from Nakai-Moishezon.

Suppose that $D$ is nef. By induction on $n = \dim X$ it suffices to prove that

$$D^n \geq 0.$$

Pick an ample divisor $H$. Then $D + tH$ is nef for all $t \geq 0$. We have

$$f(t) = (D + tH)^n = \sum \binom{n}{i} D^i H^{n-i} t^n t^{-i},$$

is a polynomial in $t$, all of whose terms are non-negative, except maybe the constant term, which tends to infinity as $t$ tends to infinity. Suppose that $D^n \leq 0$. Then there is a real number $t_0 \in [0, \infty)$ such that

$$f(t_0) = 0,$$

and $f(t) > 0$ for all $t > t_0$. Pick $t > t_0$ rational. Then $D + tH$ is ample, by Nakai’s criteria. In particular we may find a divisor $B \in |k(D + tH)|$ for some positive integer $k$. We may write

$$f(t) = t^n H^n + \sum \binom{n-1}{i} t^i (H + tD)^{n-i-1} D.$$

Consider the product $H^i (H + tD)^{n-i-1}$. Pick $k$ such that $kH$ is very ample and pick $l$ such that $l(H + tD)$ is very ample. Pick general elements $H_1, H_2, \ldots, H_i \in |kH|$ and $G_1, G_2, \ldots, G_{n-i-1} \in |l(H + tD)|$. Then the intersection

$$C = H_1 \cdot H_2 \cdots H_i \cdot G_1 \cdot G_2 \cdots G_{n-i-1} \equiv \frac{1}{kil^{n-i-1}} H^i \cdot (H + tD)^{n-i-1},$$

is a smooth curve (here $\equiv$ denotes numerical equivalence, meaning that both sides dot with any Cartier divisor the same). Thus every term is non-negative, as $D \cdot C \geq 0$. But then

$$0 = f(t_0) = \lim_{t \to t_0} f(t) \geq t_0^n H^n.$$

Thus $t_0 = 0$, and $D^n \geq 0$.

Lemma 6.5. Let $X$ be a normal projective variety and let $\pi: Y \to X$ blow up a smooth point $p$ of $X$.

Then $E^n = (-1)^{n-1}$. 

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Proof. Since this result is local in the analytic topology, we may as
well assume that $X = \mathbb{P}^n$. Choose coordinates $x_1, x_2, \ldots, x_n$ about
the point $p$. Then coordinates on $Y \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ are given by the equations

$$x_iY_j = x_jY_i.$$  

(These equations simply express the fact that $(x_1, x_2, \ldots, x_n)$ defines
the point $[Y_1 : Y_2 : \cdots : Y_n] \in \mathbb{P}^{n-1}$.) On the coordinate chart $Y_n \neq 0,$
we have affine coordinates $y_i = Y_i/Y_n$ on $\mathbb{P}^{n-1}$ and since

$$x_i = x_ny_i,$$

it follows that $x_n, y_1, y_2, \ldots, y_{n-1}$ are coordinates on $Y$, and the ex-
ceptional divisor is given locally by $x_n = 0$. Let $H$ be the class of a
hyperplane in $\mathbb{P}^n$ which passes through $p$. Then we may assume that
$H$ is given by $x_1 = 0$. Since $x_1 = x_ny_1$ it follows that

$$\pi^*H = G + E,$$

where $G$ defined by $y_1 = 0$, is the strict transform of $H$. Now $G|_E$
restricts to a hyperplane in $E$. Thus

$$E|_E = -G|_E,$$

since $E$ pushes forward to zero. But then

$$E^n = (E|_E)^{n-1} = (-1)^{n-1}.$$  

□

Definition-Lemma 6.6 (Kodaira’s Lemma). Let $X$ be a normal pro-
jective variety of dimension $n$ and let $D$ be a $\mathbb{Q}$-Cartier divisor.

TFAE

(1) $h^0(X, \mathcal{O}_X(mD)) > \alpha m^n$, for some constant $\alpha > 0$, for any $m$
which is sufficiently divisible.

(2) $D \sim_\mathbb{Q} A + E$, where $A$ is an ample divisor and $E \geq 0$.

If further $D$ is nef then these conditions are equivalent to

(3) $D^n > 0$.

If any of these conditions hold we say that $D$ is **big**.

Proof. Let $H \geq 0$ be any ample Cartier divisor. If $m$ is sufficiently
large, then

$$h^i(X, \mathcal{O}_X(mH)) = 0,$$

so that by Asymptotic Riemann Roch there are positive constants $\alpha_i$
such that

$$\alpha_1m^n < h^0(X, \mathcal{O}_X(mH)) < \alpha_2m^n,$$

for all $m$. Now let $G$ be any divisor. Pick $k > 0$ such that $G + kH$ is
ample. Then

$$h^0(X, \mathcal{O}_X(mG)) \leq h^0(X, \mathcal{O}_X(mG + mkH)) \leq \beta m^n,$$
for some constant $\beta_1$.

Suppose that (1) holds. Let $H$ be an ample Cartier divisor. Then there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(mD - H) \rightarrow \mathcal{O}_X(mD) \rightarrow \mathcal{O}_H(mD) \rightarrow 0.$$  

Now

$$h^0(H, \mathcal{O}_H(mD)) \leq \beta m^{n-1},$$

for some constant $\beta$. It follows that

$$h^0(X, \mathcal{O}_X(mD - H)) > \alpha m^n.$$

In particular we may find $B$ such that

$$B \in |mD - H|.$$  

But then

$$D \sim_{\mathbb{Q}} H/m + B/m = A + E.$$  

Thus (1) implies (2).

How suppose that (2) holds. Replacing $D$ by a multiple, we may assume that $D \sim A + E$. But then

$$h^0(X, \mathcal{O}_X(mD)) \geq h^0(X, \mathcal{O}_X(mA)) > \alpha m^n,$$

for some constant $\alpha > 0$. Thus (2) implies (1).

Now suppose that $D$ is nef. Assume that (2) holds. We may assume that $A$ is very ample and a general element of $|A|$. Then

$$D^n = A \cdot D^{n-1} + E \cdot D^{n-1} \geq (D|_A)^{n-1} > 0,$$

by induction on the dimension. Thus (2) implies (3).

Finally suppose that (3) holds. Let $\pi: Y \rightarrow X$ be a birational morphism such that $Y$ is smooth. Since $G = \pi^* D$ is nef, $G^n = D^n$ and

$$h^0(Y, \mathcal{O}_Y(mG)) = h^0(X, \mathcal{O}_X(mD)),$$

replacing $X$ by $Y$ and $D$ by $G$, we may assume that $X$ is smooth. Pick a very ample divisor $H$, a general element of $|H|$, such that $H + K_X$ is also very ample and let $G \in |H + K_X|$ be a general element. There is an exact sequence

$$0 \rightarrow \mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD + G) \rightarrow \mathcal{O}_G(mD + G) \rightarrow 0.$$  

Now

$$h^i(X, \mathcal{O}_X(mD + G)) = h^i(X, \mathcal{O}_X(K_X + H + mD)) = 0$$

for $i > 0$ and $m \geq 0$ by Kodaira vanishing, as $H + mD$ is ample. Thus

$$\chi(X, \mathcal{O}_X(mD + G)) > \alpha m^n,$$
for some constant $\alpha > 0$. Since
\[ h^0(G, \mathcal{O}_G(mD + G)) < \beta m^{n-1}, \]
for some constant $\beta$, (3) implies (1). \hfill \square

**Theorem 6.7 (Seshadri’s criteria).** Let $X$ be a normal projective variety and let $D$ be a $\mathbb{Q}$-divisor.

TFAE

(1) $D$ is ample.
(2) For every point $x \in X$, there is a positive constant $\epsilon = \epsilon(x) > 0$ such that for every curve $C$,
\[ D \cdot C > \epsilon \text{mult}_x C, \]
where $\text{mult}_x C$ is the multiplicity of the point $x$ on $C$.

Proof. Suppose that $D$ is ample. Then $mD$ is very ample for some positive integer $m$. Let $C$ be a curve with a point $x$ of multiplicity $k$. Pick $y$ any other point of $C$. Then we may find $H \in |mD|$ containing $x$ and not containing $y$. In this case
\[ (mD) \cdot C = H \cdot C \geq k, \]
so that $\epsilon = 1/m$ will do. Thus (1) implies (2).

Now assume that (2) holds. We check the hypotheses for Nakai’s criteria. By induction on the dimension $n$ of $X$ it suffices to check that $D^n > 0$. Let $\pi: Y \to X$ be the blow up of $X$ at $x$, a smooth point of $X$, with exceptional divisor $E$. Consider $\pi^*D - \eta E$, for any $0 < \eta < \epsilon$. Let $\Sigma \subset Y$ be any curve on $Y$. If $\Sigma$ is contained in $E$, then
\[ (\pi^*D - \eta E) \cdot \Sigma = -E \cdot \Sigma > 0. \]
Otherwise let $C$ be the image of $\Sigma$. If the multiplicity of $C$ at $x$ is $m$, then $E \cdot \Sigma = m$. Thus
\[ (\pi^*D - \eta E) \cdot \Sigma = D \cdot C - \eta m > 0, \]
by definition of $\epsilon$. It follows that $\pi^*D - \eta E$ is nef and so $\pi^*D - \epsilon E$ is nef. By (6.4) it follows that the polynomial
\[ f(t) = (\pi^*D - tE)^n, \]
of degree $n$ in $t$ is non-negative. On the other hand, note that $E^n = \pm 1 \neq 0$. Thus the polynomial $f(t)$ is not constant. Thus $f(\eta) > 0$, some $0 < \eta < \epsilon$. It follows that
\[ h^0(X, \mathcal{O}_X(mD)) \geq h^0(Y, \mathcal{O}_X(m\pi^*D - m\eta E)) > 0, \]
for $m$ sufficiently large and divisible. It follows easily that $D^n > 0$. \hfill \square
One of the most interesting aspects of Seshadri’s criteria is that gives a local measure of ampleness:

**Definition 6.8.** Let $X$ be a normal variety, and let $D$ be a nef $\mathbb{Q}$-Cartier divisor. Given a point $x \in X$, let $\pi : Y \to X$ be the blow up of $X$ at $x$. The real number

$$\epsilon(D, x) = \inf \{ \epsilon \mid \pi^* D - \epsilon E \text{ is nef} \}$$

is called the **Seshadri constant** of $D$ at $x$.

It seems to be next to impossible to calculate the Seshadri constant in any interesting cases. For example there is no known example of a smooth surface $S$ and a point $x \in S$ such that the Seshadri constant is irrational, although this is conjectured to happen nearly all the time. One of the first interesting cases is a very general smooth quintic surface $S$ in $\mathbb{P}^3$ (so that $S$ belongs to the complement of a countable union of closed subsets of the space of all quintics $\mathbb{P}^{55}$). Suppose that $p \in S$ is a very general point. Let $\pi : T \to S$ blow up the point $p$. As $S$ is very general,

$$\text{Pic}(T) = \mathbb{Z}[\pi^* H] \oplus \mathbb{Z}[E],$$

where $H$ is the class of a hyperplane and $E$ is the exceptional divisor. Since $p$ is very general, it seems reasonable to expect that the only curve of negative self-intersection on $T$ is $E$. If this is the case then $\pi^* H - aE$ is nef if and only if its self-intersection is non-negative. Now

$$0 = (\pi^* H - aE)^2 = H^2 - a^2 = 5 - a^2.$$  

So if there are no curves of negative self-intersection other than $E$, then the Seshadri constant is $\sqrt{5}$. 

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