6. Ampleness criteria

We return to the problem of determining when a line bundle is ample.

Theorem 6.1 (Nakai-Moishezon)**.** *Let X be a normal projective variety and let D be a $\mathbb{Q}$-Cartier divisor.*

TFAE

(1) D is ample.

(2) For every subvariety $V \subset X$ of dimension k,

$$D^k \cdot V > 0.$$

Proof. Suppose that D is ample. Then mD is very ample for some $m > 0$. Let $\phi\colon X \longrightarrow \mathbb{P}^N$ be the corresponding embedding. Then $mD = \phi^* H$, where H is a hyperplane in $\mathbb{P}^N$. Then

$$D^k \cdot V = \frac{1}{m^k} H^k \cdot \phi(V) > 0,$$

since intersecting $\phi(V)$ with H^k corresponds to intersecting V with a linear space of dimension $N-k$. But this is nothing more than the degree of $\phi(V)$ in projective space.

Now suppose that D satisfies (2). Let H be a general element of a very ample linear system. Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(pD+(q-1)H) \longrightarrow \mathcal{O}_X(pD+qH) \longrightarrow \mathcal{O}_H(pD+qH) \longrightarrow 0.$$

By induction, $D|_H$ is ample. It is straightforward to prove that

$$h^i(H, \mathcal{O}_H(pD+qH)) = 0,$$

for $i > 0$, p sufficiently large and any $q > 0$, by induction on the dimension. In particular,

$$h^i(X, \mathcal{O}_X(pD+(q-1)H)) = h^i(X, \mathcal{O}_X(pD+qH)),$$

for $i > 1$, p sufficiently large and any $q \geq 1$. By Serre vanishing the last group vanishes for q sufficiently large. Thus by descending induction

$$h^i(X, \mathcal{O}_X(pD+qH)) = 0,$$

for all $q \geq 0$. Thus by (4.1) it follows that

$$h^0(X, \mathcal{O}_X(mD)) \neq 0,$$

for m sufficiently large, that is, $|mD|$ is non-empty. As usual, this means that we may assume that $D \geq 0$ is Cartier. Let $\nu\colon \tilde{D} \longrightarrow D_{\mathrm{red}}$ be the normalisation of D_{red}, the reduced subscheme associated to D. Then $\nu^* D|_{D_{\mathrm{red}}}$ is ample by induction. It follows by (3.4) that $D|_D$ is ample.

I claim that the map

$$\rho_m\colon H^0(X,\mathcal{O}_X(mD)) \longrightarrow H^0(D,\mathcal{O}_D(mD)),$$

is surjective for m sufficiently large. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0.$$

As $D|_D$ is ample,

$$h^i(D,\mathcal{O}_D(mD)) = 0,$$

for $i > 0$ and m sufficiently large, by Serre vanishing. Thus

$$h^1(X,\mathcal{O}_X(mD)) \leq h^1(X,\mathcal{O}_X((m-1)D)) \qquad \text{and} \qquad h^i(X,\mathcal{O}_X(mD)) = 0,$$

for $i > 1$, with equality if and only if ρ_m is surjective. Since

$$h^1(X,\mathcal{O}_X(mD)),$$

is finite dimensional, its dimension cannot drop infinitely often, and so ρ_m is surjective as claimed.

As $D|_D$ is ample, $(mD)|_D$ is very ample. As we can lift sections, it follows that $|mD|$ is base point free, that is, D is semiample. Let $\phi = \phi_{mD}\colon X \longrightarrow \mathbb{P}^N$ be the corresponding morphism. Then $D = \phi^*H$. Suppose that C is a curve contracted by ϕ. Then

$$D\cdot C = \phi^*H\cdot C = H\cdot \phi_*C = 0,$$

a contradiction. But then ϕ_{mD} is a finite morphism and $D = \phi^*H$ is ample by (3.4). □

Definition 6.2. *Let X be a normal projective variety and let D be a $\mathbb{Q}$-Cartier divisor. We say that D is **nef** if $D\cdot C \geq 0$ for all curves $C \subset X$.*

Lemma 6.3. *Let X be a normal variety and let D be a $\mathbb{Q}$-Cartier divisor.*

If D is semiample then D is nef.

Proof. By assumption there is a morphism $\phi\colon X \longrightarrow Y \subset \mathbb{P}^n$ such that

$$mD = \phi^*H.$$

But then

$$D\cdot C = \frac{1}{m}\phi^*C\cdot H \geq 0.$$

□

Lemma 6.4. *Let X be a normal projective variety and let D be a $\mathbb{Q}$-Cartier divisor.*

TFAE

(1) $D + H$ is ample for any ample $\mathbb{Q}$-divisor H.

(2) *If* $V \subset X$ *is any subvariety of* X *then*

$$D^k \cdot V \geq 0,$$

where V *has dimension* k.

(3) D *is nef.*

Proof. (1) implies (2) and (2) implies (3) are clear. (2) implies (1) follows from Nakai-Moishezon.

Suppose that D is nef. By induction on $n = \dim X$ it suffices to prove that

$$D^n \geq 0.$$

Pick an ample divisor H. Then $D + tH$ is nef for all $t \geq 0$. We have

$$f(t) = (D + tH)^n = \sum \binom{n}{i} D^i H^{n-i} t^{n-i},$$

is a polynomial in t, all of whose terms are non-negative, except maybe the constant term, which tends to infinity as t tends to infinity. Suppose that $D^n \leq 0$. Then there is a real number $t_0 \in [0, \infty)$ such that

$$f(t_0) = 0,$$

and $f(t) > 0$ for all $t > t_0$. Pick $t > t_0$ rational. Then $D + tH$ is ample, by Nakai's criteria. In particular we may find a divisor $B \in |k(D+tH)|$ for some positive integer k. We may write

$$f(t) = t^n H^n + \sum \binom{n-1}{i} t^i H^i (H + tD)^{n-i-1} D.$$

Consider the product $H^i(H + tD)^{n-i-1}$. Pick k such that kH is very ample and pick l such that $l(H + tD)$ is very ample. Pick general elements $H_1, H_2, \dots, H_i \in |kH|$ and $G_1, G_2, \dots, G_{n-i-1} \in |l(H + tD)|$. Then the intersection

$$C = H_1 \cdot H_2 \cdots H_i \cdot G_1 \cdot G_2 \cdots G_{n-i-1} \equiv \frac{1}{k^i l^{n-i-1}} H^i \cdot (H + tD)^{n-i-1},$$

is a smooth curve (here $\equiv$ denotes numerical equivalence, meaning that both sides dot with any Cartier divisor the same). Thus every term is non-negative, as $D \cdot C \geq 0$. But then

$$0 = f(t_0) = \lim_{t \to t_0} f(t) \geq t_0^n H^n.$$

Thus $t_0 = 0$, and $D^n \geq 0$. □

Lemma 6.5. *Let* X *be a normal projective variety and let* $\pi \colon Y \longrightarrow X$ *blow up a smooth point* p *of* X.

Then $E^n = (-1)^{n-1}$.

Proof. Since this result is local in the analytic topology, we may as well assume that $X = \mathbb{P}^n$. Choose coordinatees $x_1, x_2, \dots, x_n$ about the point p. Then coordinates on $Y \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ are given by the equations

$$x_i Y_j = x_j Y_i.$$

(These equations simply express the fact that $(x_1, x_2, \dots, x_n)$ defines the point $[Y_1 : Y_2 : \cdots : Y_n] \in \mathbb{P}^{n-1}$.). On the coordinate chart $Y_n \neq 0$, we have affine coordinates $y_i = Y_i/Y_n$ on $\mathbb{P}^{n-1}$ and since

$$x_i = x_n y_i,$$

it follows that x_n, $y_1, y_2, \dots, y_{n-1}$ are coordinates on Y, and the exceptional divisor is given locally by $x_n = 0$. Let H be the class of a hyperplane in $\mathbb{P}^n$ which passes through p. Then we may assume that H is given by $x_1 = 0$. Since $x_1 = x_n y_1$ it follows that

$$\pi^* H = G + E,$$

where G defined by $y_1 = 0$, is the strict transform of H. Now $G|_E$ restricts to a hyperplane in E. Thus

$$E|_E = -G|_E,$$

since E pushes forward to zero. But then

$$E^n = (E|_E)^{n-1} = (-1)^{n-1}. \qquad \square$$

Definition-Lemma 6.6 (Kodaira's Lemma)**.** *Let X be a normal projective variety of dimension n and let D be a $\mathbb{Q}$-Cartier divisor.*

TFAE

(1) $h^0(X, \mathcal{O}_X(mD)) > \alpha m^n$, for some constant $\alpha > 0$, for any m which is sufficiently divisible.

(2) $D \sim_{\mathbb{Q}} A + E$, where A is an ample divisor and $E \geq 0$.

If further D is nef then these conditions are equivalent to

(3) $D^n > 0$.

If any of these conditions hold we say that D is ***big****.*

Proof. Let $H \geq 0$ be any ample Cartier divisor. If m is sufficiently large, then

$$h^i(X, \mathcal{O}_X(mH)) = 0,$$

so that by Asymptotic Riemann Roch there are positive constants α_i such that

$$\alpha_1 m^n < h^0(X, \mathcal{O}_X(mH)) < \alpha_2 m^n,$$

for all m. Now let G be any divisor. Pick $k > 0$ such that $G + kH$ is ample. Then

$$h^0(X, \mathcal{O}_X(mG)) \leq h^0(X, \mathcal{O}_X(mG + mkH)) \leq \beta_1 m^n,$$

for some constant β_1.

Suppose that (1) holds. Let H be an ample Cartier divisor. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD - H) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_H(mD) \longrightarrow 0.$$

Now

$$h^0(H, \mathcal{O}_H(mD)) \leq \beta m^{n-1},$$

for some constant β. It follows that

$$h^0(X, \mathcal{O}_X(mD - H)) > \alpha m^n.$$

In particular we may find B such that

$$B \in |mD - H|.$$

But then

$$D \sim_{\mathbb{Q}} H/m + B/m = A + E.$$

Thus (1) implies (2).

How suppose that (2) holds. Replacing D by a multiple, we may assume that $D \sim A + E$. But then

$$h^0(X, \mathcal{O}_X(mD)) \geq h^0(X, \mathcal{O}_X(mA)) > \alpha m^n,$$

for some constant $\alpha > 0$. Thus (2) implies (1).

Now suppose that D is nef. Assume that (2) holds. We may assume that A is very ample and a general element of $|A|$. Then

$$D^n = A \cdot D^{n-1} + E \cdot D^{n-1} \geq (D|_A)^{n-1} > 0,$$

by induction on the dimension. Thus (2) implies (3).

Finally suppose that (3) holds. Let $\pi\colon Y \longrightarrow X$ be a birational morphism such that Y is smooth. Since $G = \pi^* D$ is nef, $G^n = D^n$ and

$$h^0(Y, \mathcal{O}_Y(mG)) = h^0(X, \mathcal{O}_X(mD)),$$

replacing X by Y and D by G, we may assume that X is smooth. Pick a very ample divisor H, a general element of $|H|$, such that $H + K_X$ is also very ample and let $G \in |H + K_X|$ be a general element. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_X(mD + G) \longrightarrow \mathcal{O}_G(mD + G) \longrightarrow 0.$$

Now

$$h^i(X, \mathcal{O}_X(mD + G)) = h^i(X, \mathcal{O}_X(K_X + H + mD)) = 0$$

for $i > 0$ and $m \geq 0$ by Kodaira vanishing, as $H + mD$ is ample. Thus

$$\chi(X, \mathcal{O}_X(mD + G)) > \alpha m^n,$$

for some constant $\alpha > 0$. Since

$$h^0(G, \mathcal{O}_G(mD+G)) < \beta m^{n-1},$$

for some constant β, (3) implies (1). □

Theorem 6.7 (Seshadri's criteria)**.** *Let X be a normal projective variety and let D be a $\mathbb{Q}$-divisor.*

TFAE

(1) D is ample.

(2) For every point $x \in X$, there is a positive constant $\epsilon = \epsilon(x) > 0$ such that for every curve C,

$$D \cdot C > \epsilon \operatorname{mult}_x C,$$

where $\operatorname{mult}_x C$ is the multiplicity of the point x on C.

Proof. Suppose that D is ample. Then mD is very ample for some positive integer m. Let C be a curve with a point x of multiplicity k. Pick y any other point of C. Then we may find $H \in |mD|$ containing x and not containing y. In this case

$$(mD) \cdot C = H \cdot C \geq k,$$

so that $\epsilon = 1/m$ will do. Thus (1) implies (2).

Now assume that (2) holds. We check the hypotheses for Nakai's criteria. By induction on the dimension n of X it suffices to check that $D^n > 0$. Let $\pi \colon Y \longrightarrow X$ be the blow up of X at x, a smooth point of X, with exceptional divisor E. Consider $\pi^*D - \eta E$, for any $0 < \eta < \epsilon$. Let $\Sigma \subset Y$ be any curve on Y. If Σ is contained in E, then

$$(\pi^*D - \eta E) \cdot \Sigma = -E \cdot \Sigma > 0.$$

Otherwise let C be the image of Σ. If the multiplicity of C at x is m, then $E \cdot \Sigma = m$. Thus

$$(\pi^*D - \eta E) \cdot \Sigma = D \cdot C - \eta m > 0,$$

by definition of ϵ. It follows that $\pi^*D - \eta E$ is nef and so $\pi^*D - \epsilon E$ is nef. By (6.4) it follows that the polynomial

$$f(t) = (\pi^*D - tE)^n,$$

of degree n in t is non-negative. On the other hand, note that $E^n = \pm 1 \neq 0$. Thus the polynomial $f(t)$ is not constant. Thus $f(\eta) > 0$, some $0 < \eta < \epsilon$. It follows that

$$h^0(X, \mathcal{O}_X(mD)) \geq h^0(Y, \mathcal{O}_X(m\pi^*D - m\eta E)) > 0,$$

for m sufficiently large and divisible. It follows easily that $D^n > 0$. □

One of the most interesting aspects of Seshadri's criteria is that gives a local measure of ampleness:

Definition 6.8. *Let X be a normal variety, and let D be a nef $\mathbb{Q}$-Cartier divisor. Given a point $x \in X$, let $\pi\colon Y \longrightarrow X$ be the blow up of X at x. The real number*

$$\epsilon(D, x) = \inf\{\, \epsilon \,|\, \pi^* D - \epsilon E \text{ is nef}\}$$

is called the ***Seshadri constant*** *of D at x.*

It seems to be next to impossible to calculate the Seshadri contant in any interesting cases. For example there is no known example of a smooth surface S and a point $x \in S$ such that the Seshadri constant is irrational, although this is conjectured to happen nearly all the time. One of the first interesting cases is a very general smooth quintic surface S in $\mathbb{P}^3$ (so that S belongs to the complement of a countable union of closed subsets of the space of all quintics $\mathbb{P}^{55}$). Suppose that $p \in S$ is a very general point . Let $\pi\colon T \longrightarrow S$ blow up the point p. As S is very general,

$$\operatorname{Pic}(T) = \mathbb{Z}[\pi^* H] \oplus \mathbb{Z}[E],$$

where H is the class of a hyperplane and E is the exceptional divisor. Since p is very general, it seems reasonable to expect that the only curve of negative self-intersection on T is E. If this is the case then $\pi^* H - aE$ is nef if and only if its self-intersection is non-negative. Now

$$0 = (\pi^* H - aE)^2 = H^2 - a^2 = 5 - a^2.$$

So if there are no curves of negative self-intersection other than E, then the Seshadri constant is $\sqrt{5}$.