7. Closed cone of curves

Finally let us turn to one of the most interesting ampleness criteria.

Definition 7.1. *Let X be a smooth projective variety. The* ***cone of curves*** *of X, denoted $\mathrm{NE}(X)$, is the cone spanned by the classes $[C] \in H_2(X, \mathbb{R})$ inside the vector space $H_2(X, \mathbb{R})$. The* ***closed cone of curves*** *of X, denoted $\overline{\mathrm{NE}}(X)$, is the closure of $\mathrm{NE}(X)$ inside $H_2(X, \mathbb{R})$.*

More generally, if X is normal but not necessarily smooth, one can define the cone of curves as the cone in the real vector space of curves modulo numerical equivalence.

The significance of the closed cone of curves is given by:

Theorem 7.2 (Kleiman's criterion)**.** *Let X be a normal projective variety.*

TFAE:

1. *D is ample.*
2. *D defines a positive linear functional on*

$$\overline{\mathrm{NE}}(X) - \{0\} \longrightarrow \mathbb{R}^+$$

defined by extending the map $[C] \longrightarrow H \cdot C$ linearly.

In particular $\overline{\mathrm{NE}}(X)$ does not contain a line and if H is ample then the set

$$\{\, \alpha \in \overline{\mathrm{NE}}(X) \,|\, H \cdot C \leq k \,\},$$

is compact, where k is any positive constant.

Proof. Suppose that H is ample. Then H is certainly positive on $\mathrm{NE}(X)$. Suppose that it is not positive on $\overline{\mathrm{NE}}(X)$. Then there is a non-zero class $\alpha \in \overline{\mathrm{NE}}(X)$ such that $H \cdot \alpha = 0$, which is a limit of classes α_i, where $\alpha_i = \sum a_{ij}[C_{ij}]$, $a_{ij} \geq 0$ are positive real numbers. Pick a $\mathbb{Q}$-Cartier divisor M such that $M \cdot \alpha < 0$. Then there is a positive integer m such that $mH + M$ is ample. But then

$$0 > (mH + M) \cdot \alpha = \lim_i (mH + M) \cdot \alpha_i \geq 0,$$

a contradiction. Thus (1) implies (2).

Suppose that $\overline{\mathrm{NE}}(X)$ contains a line. Then it contains a line through the origin, which is impossible, since X contains an ample divisor H, which is positive on this line outside the origin. Pick a basis $M_1, M_2, \dots, M_k$ for the the space of Cartier divisors modulo numerical equivalence, which is the vector space dual to the space spanned by $\overline{\mathrm{NE}}(X)$. Then we may pick m such that $mH \pm M_i$ is ample for all $1 \leq i \leq k$. In particular

$$|M_i \cdot \alpha| \leq mH \cdot \alpha.$$

Thus the set above is compact as it is a closed subset of a big cube.

Now suppose that (2) holds. By what we just proved, the set of $\alpha \in \overline{\mathrm{NE}}(X)$ such that $H \cdot \alpha = 1$ forms a compact slice of $\overline{\mathrm{NE}}(X)$ (meaning that this set is compact and $\overline{\mathrm{NE}}(X)$ is the cone over this convex set). Thus $B = D - \epsilon H$ is nef, for some $\epsilon > 0$. But then

$$D = (D - \epsilon H) + \epsilon H = B + \epsilon H,$$

is ample, by (6.4) and Nakai's criterion. □

Mori realised that Kleiman's criteria presents a straightforward way to classify projective varieties.

Definition 7.3. *Let $f \colon X \longrightarrow Y$ be a morphism of varieties. We say that f is a* ***contraction*** *morphism if $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

A contraction morphism always has connected fibres; if Y is normal and f has connected fibres then f is a contraction morphism. Consider the category of projective varieties and contraction morphisms. Any morphism $f \colon X \longrightarrow Y$ is determined by the curves contracted by f. Indeed let $\sim$ be the equivalence relation on the points of X, generated by declaring two points to be equivalent if they both belong to an irreducible curve contracted by f. Thus $x \sim y$ if and only if x and y belong to a connected curve C which is contracted to a point.

I claim that $x \sim y$ determines f. First observe that $x \sim y$ if and only if $f(x) = f(y)$. It is clear that if $x \sim y$ then $f(x) = f(y)$. Conversely suppose that $f(x) = f(y) = p$. If $d = \dim f^{-1}(p) \leq 1$ there is nothing to prove. If $d > 1$ then since the fibres of f are connected, we may assume that x and y belong to the same irreducible component. Pick an ample divisor $H \subset f^{-1}(p)$ containing both x and y. Then $\dim H = d - 1$. Repeating this operation we find a curve containing by x and y. This determines $Y = X/\sim$ as a topological space, and the condition $\mathcal{O}_Y = f_*\mathcal{O}_X$ determines the scheme structure.

Note also that there is then a partial correspondence

(1) faces of the Mori cone $\overline{\mathrm{NE}}(X)$.
(2) contraction morphisms $\phi \colon X \longrightarrow Y$.

Given a contraction morphism ϕ, let

$$F = \{\, \alpha \in \overline{\mathrm{NE}}(X) \,|\, D \cdot \alpha = 0 \,\},$$

where $D = \phi^* H$. Kleimans' criteria then says that F is a face. The problem is that given a face F of the closed cone it is in general impossible to contract F. For a start F must be rational (that is, spanned by integral classes).

Equivalently and dually, the problem is that there are nef divisors which are not semiample.