MODEL ANSWERS TO THE FIRST HOMEWORK

1. It remains to check that ω has a simple pole at infinity. There are two ways to proceed.

One is by direct computation (alas). By definition

$$x_i = \frac{X_i}{X_0},$$

for $1 \leq i \leq n$. Consider coordinates

$$y_j = \frac{X_j}{X_n}.$$

for $0 \leq j \leq n-1$. Then we have the coordinate transformation

$$x_i = y_i y_0,$$

for $1 \leq i \leq n-1$ and

$$x_n = y_0^{-1}.$$

Then

$$\frac{\mathrm{d}x_i}{x_i} = \frac{\mathrm{d}(y_i y_0)}{y_i y_0} = \frac{\mathrm{d}y_i}{y_i} + \frac{\mathrm{d}y_0}{y_0},$$

and

$$\frac{\mathrm{d}x_n}{x_n} = -\frac{\mathrm{d}y_0}{y_0}.$$

Thus

$$\omega = -\frac{\mathrm{d}y_1}{y_1} \wedge \frac{\mathrm{d}y_2}{y_2} \wedge \cdots \wedge \frac{\mathrm{d}y_0}{y_0},$$

and the result follows.

Aliter Alternatively note that it suffices to prove that $-K_X = -(n+1)H$. We know that $K_X = -kH$ for some k and we just want to show $k = n+1$. For this we proceed by induction on n. The case $n = 1$ we get by direct computation. Otherwise, let's apply adjunction to a hyperplane H.

$$K_H = (K_{\mathbb{P}^n} + H)|_H = (1-k)H|_H.$$

But $H \simeq \mathbb{P}^{n-1}$ so that $K_H = -nH|_H$ by induction. It follows that $k - 1 = n$ so that $k = n+1$.

2. By Serre duality

$$h^0(C, \mathcal{O}_C(K_C)) = h^1(C, \mathcal{O}_C) = g.$$

By Riemann-Roch applied to K_C, we have

$$\chi(C, \mathcal{O}_C(K_C)) = d - g + 1,$$

where $d = \deg K_C$. Now

$$h^1(C, \mathcal{O}_C(K_C)) = h^0(C, \mathcal{O}_C) = 1,$$

by Serre duality. Thus

$$d = g - 1 + g - 1 = 2g - 2.$$

3. By adjunction

$$K_C = (K_{\mathbb{P}^2} + C)|_C = (d-3)H|_C,$$

where H is the class of line. Taking degrees, we get

$$2g - 2 = d(d-3).$$

Solving for d yields

$$g = \frac{d(d-3)}{2} + 1 = \frac{(d-1)(d-2)}{2}.$$

4. We have a morphism $\pi\colon C \longrightarrow \mathbb{P}^1$. Since a general line meets C in d points, the degree of π is d. By Riemman-Hurwitz we have

$$2g - 2 = d(2h - 2) + b,$$

where $g = g(C)$, $h = g(\mathbb{P}^1)$. Now

$$g = \binom{d-1}{2} \qquad \text{and} \qquad h = 0,$$

and b counts the number of branch points, as the ramification is simple. Thus

$$d(d-3) = -2d + b \qquad \text{so that} \qquad b = d(d-1).$$

5. If

$$(C, \Delta) = (C, \sum_i \frac{r_i - 1}{r_i} p_i),$$

then

$$d = \deg(K_C + \Delta) = (2g-2) + \sum_i \frac{r_i - 1}{r_i},$$

so the problem is simply a problem in combinatorics. The possible cases are

- $g = 1$, Δ empty. In this case $d = 0$.
- $g = 0$, the support of Δ is at most two points. $d = 0$ iff we have exactly two points of coefficient one.

- $g = 0$, and there are three points, with coefficients
$$(\frac{p-1}{p}, \frac{q-1}{q}, \frac{r-1}{r}).$$
In this case the possibilities for (p, q, r) are
$$(2, 2, r), (2, 2, \infty), (2, 3, 3), (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 4), (3, 3, 3).$$
In the second and last three cases we have $d = 0$.
- $g = 0$ and there are four points. In this case the coefficients are
$$(1/2, 1/2, 1/2, 1/2),$$
and $d = 0$.

6. We simply have to minimise
$$d = \deg(K_C + \Delta) = (2g - 2) + \sum_i \frac{r_i - 1}{r_i},$$
subject to the constraint that $d > 0$. If $g \geq 2$ the best we can do is take Δ empty, in which case $d = 2$. If $g = 1$ then the best we can do is take $\Delta = p/2$, in which case $d = 1/2$. Suppose $g = 0$. Take the list of triples or quadruples and trying increasing one of p, q, r or s. If we increase r by one the degree increases by
$$\frac{r}{r+1} - \frac{r-1}{r} = \frac{1}{r(r+1)}.$$
In each list we should then surely increase the largest index (which is always r). Trial and error gives that the best we can do is
$$(\mathbb{P}^1, \frac{p}{2} + \frac{2q}{3} + \frac{6r}{7}),$$
in which case
$$d = \frac{1}{42} = \frac{1}{6 \cdot 7},$$
which is clearly better than the other two possibilities.

7. Let $\pi \colon B \longrightarrow C = B/G$ be the quotient morphism. Then the degree m of π is the cardinality of G. We have
$$K_C = \pi^*(K_C + \Delta).$$
Thus
$$2g - 2 = md,$$
where d is the degree of $K_C + \Delta$. Since $2g - 2 > 0$, $d > 0$ and to maximise m we should minimise d. Thus
$$m = \frac{(2g-2)}{d} \leq 84(g - 1),$$
by the previous question.

8. Consider the normalisation C of the plane curve

$$Y^7 = X^2(X - Z)Z^4.$$

Projection from the point $[0 : 1 : 0]$ defines a cover of degree seven of $\mathbb{P}^1$, totally ramified over 0, 1 and ∞. Thus

$$2g - 2 = -7 \cdot 2 + 3 \cdot 6 = 4.$$

Thus $g(C) = g = 3$. Question 7 says that the automorphism group has order at most 168.
Now this curve has an automorphism of order seven given by

$$[X : Y : Z] \longrightarrow [X : \zeta Y : Z],$$

where ζ is a primitive 7th root of unity. In fact the cover above is cyclic and is given by this automorphism. The map

$$[X : Y : Z] \longrightarrow [-X : Y : -Z],$$

defines an involution of C, which does not respect this cover. On the other hand, permutation of the branch points defines a subgroup isomorphic to S_3. Thus the order of the automorphism group is at least

$$7 \cdot 3 \cdot 4 = 84.$$

Let $f\colon C \longrightarrow B = C/G$ be the corresponding cover. It is easy to see that $B = \mathbb{P}^1$ and that there are three points of ramification. At one of the points we must have ramification of order at least 7, at another order of least 3. If the order of ramification were not 2 at the other point, then the order of the automorphism group would be too small, by the analysis given in 8 (see also 9 below).
9. Well the next triple in the list is

$$(1/2, 2/3, 7/8),$$

in which case the degree is

$$7/8 - 5/6 = (42 - 40)/48 = 1/24.$$

Thus if the order of the automorphism group is less than $84(g - 1)$ it is at most $48(g - 1)$.
10. I guess that for surfaces one should take $S = \mathbb{P}^2$. In this case, a little bit of trial and error suggests taking

$$(S, \Delta) = (\mathbb{P}^2, \frac{L_1}{2} + \frac{2L_2}{3} + \frac{6L_2}{7} + \frac{41L_1}{42}),$$

where $L_1, L_2, \dots, L_4$ are four general lines in $\mathbb{P}^2$. In this case

$$K_{\mathbb{P}^2} + \Delta = (-3 + \frac{1}{2} + \frac{2}{3} + \frac{6}{7} + \frac{41}{42})H = \frac{1}{42}H.$$

so that

$$d = (K_{\mathbb{P}^2} + \Delta)^2 = \frac{1}{(42)^2}.$$

11. I guess that for projective varieties of dimension n one should take $X = \mathbb{P}^n$. Looking at the case of curves and surfaces it is clear that Δ ought to be a sum of hyperplanes, let us suppose with coefficients

$$\frac{r_i - 1}{r_i}.$$

Looking at the case of curves and surfaces, it seems pretty clear that the choice of r_i is not a function of n, only the number we include (namely $n+2$); in other words the numbers $r_1, r_2, \dots, r_{n+2}$ are defined recursively. Note that we want

$$\sum_{i=1}^{n+2} \frac{r_i - 1}{r_i} > n + 1.$$

Thus

$$\sum_{i=1}^{n+2} 1 - \frac{1}{r_i} > n + 1,$$

that is

$$s_{n+2} = \sum_{i=1}^{n+2} \frac{1}{r_i} < 1.$$

Suppose that we have defined $r_1, r_2, \dots, r_{n+1}$. Then we choose r_{n+2} so that

$$s_{n+2} = s_{n+1} + \frac{1}{r_{n+2}} < 1,$$

that is pick r_{n+2} minimal such that

$$\frac{1}{r_{n+2}} > 1 - s_{n+2}.$$

Again, looking at the case $n = 1$, 2, we guess that

$$s_n = 1 - \frac{1}{k_n},$$

for some integer k_n. But then we should take

$$r_n = k_n + 1,$$

and we get the recursion

$$1 - \frac{1}{k_{n+1}} = 1 - \frac{1}{k_n} + \frac{1}{k_n + 1}.$$

Thus
$$\frac{1}{k_{n+1}} = \frac{1}{k_n} - \frac{1}{k_n+1} = \frac{1}{k_n(k_n+1)}.$$
Thus
$$k_{n+1} = k_n(k_n+1) \qquad \text{and} \qquad r_{n+1} = r_n(r_n-1)+1.$$
where $k_1 = 1$. Thus
$$k_1 = 1, k_2 = 2; k_3 = 6; k_4 = 42; k_5 = 42 \cdot 43, \dots,$$
with corresponding values
$$s_1 = 0, s_2 = 1/2, s_3 = 5/6, s_4 = 41/42,$$
and
$$r_1 = 2, r_2 = 3; r_3 = 7; r_4 = 43; r_5 = 42 \cdot 43 + 1, \dots.$$