MODEL ANSWERS TO THE SECOND HOMEWORK

1. $D \in |H|$ if and only if D is a divisor of degree d. Thus

$$|H| = \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))),$$

$|H|$ is base point free. Indeed, if $p = [a_0 : a_1 : \dots : a_n]$, then $a_i \neq 0$ some i. But then

$$p \notin dH_i \in |H|,$$

where H_i is the zero locus of X_i. But then p does not belong to the base locus of $|H|$. The dimension is

$$h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) - 1 = \binom{n+d}{n} - 1.$$

2. If $C = \mathbb{P}^1$, then $h^0(C, \mathcal{O}_C(p)) > 1$, either by (1) or Riemann-Roch. Suppose that $h^0(C, \mathcal{O}_C(p)) > 1$. Then the linear system $|p|$ has no base points, so that it defines a morphism

$$\phi\colon C \longrightarrow \mathbb{P}^1,$$

of degree one. But then $C \simeq \mathbb{P}^1$.
3. Suppose that $\deg D > 2g$. Then

$$h^0(C, \mathcal{O}_C(K_C - D)) = h^0(C, \mathcal{O}_C(K_C - D - p)) = h^0(C, \mathcal{O}_C(K_C - D - p - q)) = 0,$$

for all points p and q. But then by Riemman-Roch

$$h^0(C, \mathcal{O}_C(D - p)) = (d-1) - g + 1 = h^0(C, \mathcal{O}_C(D)) - 1,$$

and

$$h^0(C, \mathcal{O}_C(D - p - q)) = (d-2) - g + 1 = h^0(C, \mathcal{O}_C(D)) - 2.$$

It follows that $\phi = \phi_{|D|}$ is an embedding, so that D is very ample.
Suppose that $D = K_C + p + q$. Then

$$h^1(C, \mathcal{O}_C(D - p - q)) = h^0(C, \mathcal{O}_C) = 1.$$

Thus

$$|D - p - q| + p = |D - q|,$$

so that $\phi(p) = \phi(q)$. But then ϕ is not an embedding.
If $d > 0$ then $\deg(2g+1)D > 2g$, so that $(2g+1)D$ is very ample and D is ample. Conversely if D is ample then some multiple is linealry equivalent to an effective divisor so that $d > 0$.