MODEL ANSWERS TO THE FOURTH HOMEWORK

1. (i) Here is a pretty sneaky way to solve this problem. First note that this problem is étale local about any pair of points p and q. But any smooth curve is étale locally equivalent to $\mathbb{P}^1$. So we may assume that $C = \mathbb{P}^1$. In this case divisors of degree two can be identified with polynomials of degree two, modulo constants, that is $\mathbb{P}^2$, which is smooth.
Alternatively, working locally, we may assume that $C = \operatorname{Spec} \mathbb{C}[t]$ is affine. On $C \times C$ we have coordinates x and y, so that $C \times C = \operatorname{Spec} \mathbb{C}[x, y]$. Then $C_2 = \operatorname{Spec} \mathbb{C}[x, y]^{\mathbb{Z}_2}$. Now the action of $\mathbb{Z}_2$ is the standard one, given by swapping x and y. The ring of invariants is

$$\mathbb{C}[x, y]^{\mathbb{Z}_2} = \mathbb{C}[x + y, xy] = \mathbb{C}[u, v],$$

by a classical result, whose proof goes back to Newton. But then C_2 is smooth.
(ii) Clear.
(iii) Consider the linear system $|K_C| \subset C_2$. The Abel-Jacobi map collapses this to a point, since by definition the fibres of the Abel-Jacobi map are linear systems. By Riemann-Roch, $|K_C|$ is a g^1_2 and so defines a copy of $\mathbb{P}^1 \subset C_2$.
(iv) $\overline{\mathrm{NE}}(C_2) \subset \mathbb{R}^2$. Thus there are two rays, of which one is given by the g^1_2. The first thing to do is compute the intersection pairing between the basis elements x and δ. We will use push-pull, which we may state in the form

$$\alpha \cdot f^*\beta = f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta,$$

where $f \colon X \longrightarrow Y$ is a finite morphism of degree d, α is a cycle of dimension k on X and β is a cycle of dimension $n - k$ on Y.
We apply this to the natural map $f \colon C \times C \longrightarrow C_2$ of degree 2. First note that $f^*\delta = \Delta \subset C \times C$, where Δ is the diagonal. Now $\Delta \simeq C$, and

$$2 = K_C = (K_S + \Delta) \cdot \Delta = 2K_C + \Delta \cdot \Delta = 4 + \Delta^2.$$

Thus $\Delta^2 = -2$. But then

$$\delta^2 = f^*\delta \cdot \Delta = 2\Delta^2 = -4.$$

In particular δ generates the other extremal ray. Now note that

$$f^*x = X = X_1 + X_2 = [\{p\} \times C] + [C \times \{p\}],$$

where $p \in C$ is any point.

$$x^2 = f^*x \cdot X_1 = 1.$$

Finally,

$$x \cdot \delta = X_1 \cdot f^*\delta = 2X_1 \cdot \Delta = 2.$$

We want to calculate the class $\gamma = ax + b\delta$ of the g^1_2. We have

$$1 = \gamma \cdot x = (ax + b\delta) \cdot x = a + 2b.$$

How about $\gamma \cdot \delta$? This counts the number r of ramification points of the g^1_2. By Riemann-Hurwitz,

$$2 = 2g - 2 = -2 \cdot 2 + r = r,$$

so that $r = 6$. Thus

$$6 = \gamma \cdot \delta = (ax + b\delta) \cdot \delta = 2a - 4b.$$

Thus $a = 2$, $b = -1/2$ and $\gamma = 2x - \delta/2$. To check this calculation, note that as γ represents a copy of $\mathbb{P}^1$ contracted to a smooth point, in fact

$$-1 = \gamma^2 = (2x - \delta/2)^2 = 4x^2 - 2x \cdot \delta - \delta^2/4 = 4 - 4 - 4/4 = -1.$$

Thus $\overline{\mathrm{NE}}(C_2)$ is the two dimensional cone spanned by δ and $4x - \delta$.
(v) It is classical that the image of x represents the class θ of the Θ divisor. The Θ divisor is ample. $\pi^*\theta = x + c\gamma$, where c is determined by the constraint that

$$0 = \pi^*\theta \cdot \gamma = (x + c\gamma) \cdot \gamma = 1 - c.$$

Thus $c = 1$. The Seshadri constant is the largest d such that

$$\pi^*\theta - d\gamma$$

is nef. Since this dots γ positively, we want to find d such that

$$(\pi^*\theta - d\gamma) \cdot \delta = 0,$$

that is

$$2 - 6(d - 1) = (x - (d - 1)\gamma) \cdot \delta = 0.$$

Thus $d = 4/3$.