1. General form of Cauchys Formula

Definition 1.1. *Let U be a region. A **chain** is a formal sum of paths $\gamma_1, \gamma_2, \dots, \gamma_k$*

$$\gamma_1 + \gamma_2 + \cdots + \gamma_k,$$

where $\gamma_1, \gamma_2, \dots, \gamma_k$ are paths in U.

*A chain γ is is a **cycle** if it is a sum of closed paths.*

Note that since the integral is linear we can integrate over chains:

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z + \cdots + \int_{\gamma_k} f(z)\,\mathrm{d}z.$$

Similarly we can define the winding number of a cycle around any point in the complement of the cycle:

$$n(\gamma; a) = n(\gamma_1; a) + n(\gamma_2; a) + \cdots + n(\gamma_k; a).$$

Definition 1.2. *Let X be a topological space and let $\gamma_i \colon [0,1] \longrightarrow X$ be two paths in X, $i = 0$ and 1, such that $\gamma_i(0) = x$ and $\gamma_i(1) = y$ are both independent of i.*

*A **homotopy** from γ_0 to γ_1 is a continuous map*

$$H \colon [0,1] \times [0,1] \longrightarrow X,$$

such that $\gamma_i(t) = H(i,t)$, where $H(s,0) = x$ and $H(s,1) = y$, for all s and $t \in [0,1]$.

*We say that X is **simply connected** if any closed path in X is homotopic to a constant path.*

Intuitively this definition says that two paths are homotopic if one can be continuously deformed to the other. Fortunately it is easy to spot if a region (a connected open subset of $\mathbb{C}$) is simply connected. Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, the Riemann sphere.

Theorem 1.3. *Let U be a region.*

The following are equivalent:

1. *U is simply connected.*
2. *$\mathbb{P}^1 - U$ is connected.*
3. *$n(\gamma; a) = 0$ for all cycles in U and all points a not in U.*

Proof. We will show that (2) and (3) are equivalent and that both are implied by (1) but we skip the proof that (2) and (3) imply (1).

We know that $n(\gamma; a)$ is zero on the unbounded component of $\mathbb{C} - U$ and constant on the connected components. Thus (2) clearly implies (3). Suppose that $\mathbb{P}^1 - U = A \cup B$ is the disjoint union of two closed sets. Suppose that $\infty \in B$ so that A is bounded. Let $\delta > 0$ be the distance between A and B, the infimum of the distance between any two points $a \in A$ and $b \in B$. Tile the plane with squares of side less than $\delta/\sqrt{2}$. Pick this tiling so that $a \in A$ is the centre of a square.

Let

$$\gamma = \sum_j \partial Q_j,$$

where the sum ranges over all squares Q_j which intersect A and ∂Q_j denotes the oriented boundary of Q_j. As a is contained in precisely one square, we have

$$n(\gamma; a) = 1.$$

It is clear that γ does not meet B, by our choice of δ. On the other hand side which meets A is the side of two squares, and this sides appears with the opposite orientation on both sides. Thus if γ' is the cycle you get by cancelling these paths in γ then γ' does not meet A either.

(1) implies (3) follows from (1.4). □

Lemma 1.4. *If γ_0 and γ_1 are homotopic paths in a region U and $a \notin U$ then $n(\gamma_0; a) = n(\gamma_1; a)$.*

Proof. Let H be a homotopy from γ_0 to γ_1 and let $\gamma_s \colon [0,1] \longrightarrow U$ be the path $\gamma_s(t) = H(s,t)$. It suffices to show that $n(\gamma_s; a)$ is a continuous function of s.

We will assume that we have chosen H to be $\mathcal{C}^1$. If s_0 and $s_1 \in [0,1]$ then

$$\begin{aligned} 2\pi i(n(\gamma_{s_1}; a) - n(\gamma_{s_0}; a)) &= \int_{\gamma_{s_1}} \frac{1}{z-a}\,\mathrm{d}z - \int_{\gamma_{s_0}} \frac{1}{z-a}\,\mathrm{d}z \\ &= \int_0^1 \frac{\gamma'_{s_1}(t)}{\gamma_{s_1}(t)-a} - \frac{\gamma'_{s_0}(t)}{\gamma_{s_0}(t)-a}\,\mathrm{d}t \\ &= \int_0^1 \frac{H'(s_1,t)(H(s_0,t)-a) - H'(s_0,t)(H(s_1,t)-a)}{(H(s_1,t)-a)(H(s_0,t)-a)}\,\mathrm{d}t, \end{aligned}$$

which goes to zero as $|s_1 - s_0|$ goes to zero. □

Definition 1.5 (Cauchy's Theorem)**.** *A cycle γ in a region U is* ***homologous to zero****, with respect to U, if $n(\gamma; a) = 0$ for all points $a \in \mathbb{C} - U$.*

Theorem 1.6. *Let U be a region.*

If $f(z)$ is holomorphic on U then

$$\int_\gamma f(z)\,\mathrm{d}z = 0,$$

for every cycle γ which is homologous to zero in U.

Corollary 1.7. *If $f(z)$ is holomorphic on a simply connected region U then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0,$$

for every cycle γ in U.

Corollary 1.8. *If $f(z)$ is holomorphic and nowhere zero in a simply connected region U then it is possible to define single-valued branches of $\log f(z)$ and $\sqrt[n]{f(z)}$.*

Proof. By (1.7) we may pick a holomorphic function $F(z)$ on U such that

$$F'(z) = \frac{f'(z)}{f(z)}.$$

The derivative of the function

$$g(z) = f(z)e^{-F(z)}$$

is zero and so $g(z)$ is constant. If we pick any point $a \in U$ and one of the infinitely many possible values of $\log f(a)$, then we have

$$e^{F(z)-F(z_0)+\log f(z_0)} = f(z)$$

and so we can set

$$\log f(z) = F(z) - F(z_0) + \log f(z_0).$$

Having defined the logarithm, set

$$\sqrt[n]{f(z)} = \exp(\frac{1}{n}\log f(z)). \qquad \square$$

Proof of (1.6). We assume first that U is bounded. Tile the plane by with squares of side $\delta > 0$. Denote by Q_j, $j \in J$, those squares contained in U. As U is bounded, J is finite.

Let

$$\Gamma_\delta = \sum_{j\in J} \partial Q_j.$$

Note that Γ_δ is a sum of oriented line segments that are the sides of exactly one Q_j. Let U_δ be the interior of the union of Q_j.

Let γ be a cycle which is homologous to zero in U. Pick δ sufficiently small so that γ is contained in U_δ. Suppose that $a \in U - U_\delta$. It belongs to at least one square Q which is not one of the Q_j. Pick a point b of Q which is not in U. The line segment connecting a to b is contained in Q and so it is not contained in U_δ. Therefore a and b belong to the same connected component of $\mathbb{C} - U_\delta$ and so

$$n(\gamma; a) = n(\gamma; b) = 0,$$

by assumption.

Let f be a holomorphic function on U. If z belongs to the interior of Q_{j_0} then

$$\frac{1}{2\pi i}\int_{\partial Q_j}\frac{f(w)\,\mathrm{d}w}{w-z} = \begin{cases} f(z) & \text{if } j = j_0 \\ 0 & \text{if } j \neq j_0 \end{cases}$$

and so

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(w)\,\mathrm{d}w}{w-z}.$$

Since both sides are continuous functions of z, this holds for all $z \in U_\delta$.

Therefore

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma\left(\frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(w)\,\mathrm{d}w}{w-z}\right)\,\mathrm{d}z.$$

Since the integrand on the RHS is a continuous function of both z and w over $\Gamma_\delta \times \gamma$ it follows that we can switch the order of integration:

$$\int_\gamma\left(\frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(w)\,\mathrm{d}w}{w-z}\right)\,\mathrm{d}z = \int_{\Gamma_\delta}\left(\frac{1}{2\pi i}\int_\gamma\frac{\mathrm{d}w}{w-z}\right)f(w)\,\mathrm{d}z.$$

But the inner integral on the right is $-n(\gamma; w) = 0$. Thus

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Now suppose that U is unbounded. Let U_0 be the intersection of U with the disc $|z| < R$ where R is large enough so that γ belongs to U_0. If $a \notin U_0$ then either $a \notin U$ or a does not belong to the disc; either way $n(\gamma; a) = 0$ and so γ is homologous to zero in U_0. □