10. Riemman Mapping Theorem

Recall that a region U is by definition simply connected if every closed path is homotopic to a constant path. This is equivalent to the condition that $\mathbb{P}^1 - U$ is connected. Recall that we say that a map $f \colon U \longrightarrow V$ between two regions U and V is biholomorphic if it is a bijection and both f and its inverse are holomorphic (equivalently the derivative of f is nowhere zero). We say that a region U is proper if it is a proper subset of $\mathbb{C}$, that is, U is neither empty nor the whole of $\mathbb{C}$.

If z is a complex number then $z > 0$ means that z is real (and greater than zero).

Theorem 10.1 (Riemann Mapping Theorem)**.** *Let $U \subset \mathbb{C}$ be a simply connected proper open subset.*

Then U is biholomorphic to the interior of the unit disk, that is, there is a biholomorphic map

$$f \colon U \longrightarrow \Delta,$$

where

$$\Delta = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}.$$

Moreover if one fixes $z_0 \in U$ then there is a unique such map such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Obviously this is a key result, which is very striking. In fact this result also has some very important practical applications as well.

We first observe that uniqueness is clear. Indeed, if f_1 and f_2 are two such mappings, then the composition,

$$g = f_1 \circ f_2^{-1} \colon \Delta \longrightarrow \Delta,$$

is a map such that $g(0) = 0$ and $g'(0) > 0$. Recall that by the Schwarz Lemma an automorphism of the unit disk, which fixes the origin, is of the form $z \longrightarrow az$, for some complex number a of absolute value 1. It follows that $g(z) = z$, so that $f_1 = f_2$.

Note also that $\mathbb{C}$ is not biholomorphic to the unit disk, by Liouville's Theorem. Thus it is crucial that U is a proper subset of $\mathbb{C}$.

We now turn to existence. The idea is to consider the family

$$\mathfrak{F} = \{\, f \colon U \longrightarrow \Delta \mid f \text{ is holomorphic, injective, } f(z_0) = 0 \text{ and } f'(z_0) > 0. \,\}$$

It will turn out that the map we are looking for is the unique element whose derivative at $f'(z_0)$ is maximal. The proof of this fact has three parts,

(1) Show that $\mathfrak{F}$ is non-empty.
(2) Show that there is an element whose derivative is maximal.
(3) Show that this element has the required properties.

We start with the first and last part.

Let U be a domain and let (X, d) be a metric space. We will be interested in families (that is, sets) $\mathfrak{F}$ of continuous functions from U to X

Definition 10.2. *A family $\mathfrak{F}$ is said to be* ***normal*** *on U if every sequence of functions f_n has a subsequence which converges uniformly on every compact subset of U.*

Note one subtle part of the definition. We do not require the limit f to be an element of $\mathfrak{F}$.

Theorem 10.3. *A family $\mathfrak{F}$ of holomorphic functions is normal if and only if the functions are uniformly bounded on compact subsets.*

Recall two results:

Theorem 10.4 (Weierstrass). *Let*

$$U_1 \subset U_2 \subset U_3 \subset \dots,$$

be an infinite sequence of domains whose union is U. Suppose that $f_n(z)$ is a sequence of holomorphic function on U_n, which tends to a limit function $f(z)$ on U, uniformly on compact subsets.

Then $f(z)$ is holomorphic. Moreover $f'_n(z)$ converges uniformly on compact subsets to $f'(z)$.

Theorem 10.5 (Hurwitz). *Suppose that the holomorphic functions $f_n(z)$ converge to a function $f(z)$ on U, uniformly on compact subsets.*

If the functions $f_n(z)$ are nowhere zero then either $f(z)$ is identically zero or it is nowhere zero.

Assuming (10.3), we are now ready to complete the proof of the Riemann Mapping Theorem:

Proof of (10.1). We first show that $\mathfrak{F}$ is non-empty.

Pick a point $a \notin U$. As U is simply connected it is possible to define a single branch $h(z)$ of $\sqrt{z-a}$ on U. Then h is injective. Moreover if h takes the value w it does not take the value $-w$. By the open mapping theorem the image of h contains a disc $|w - h(z_0)| < \rho$ and so it does not intersect the disc $|w + h(z_0)| < \rho$. Equivalently, if $z \in U$ then $|h(z) + h(z_0)| \geq \rho$. In particular $|2h(z_0)| \geq \rho$.

We now check that

$$f(z) = \frac{\rho}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \frac{h(z) - h(z_0)}{h(z) + h(z_0)}$$

belongs to $\mathfrak{F}$. $f(z)$ is injective as it is the composition of $h(z)$ with a Möbius transformation. Clearly $f(z_0) = 0$.

$$f'(z_0) = \frac{\rho}{8} \frac{|h'(z_0)|}{|h(z_0)|^2} > 0.$$

On the other hand

$$\left| \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \right| = |h(z_0)| \cdot \left| \frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)} \right| \leq \frac{4|h(z_0)|}{\rho}.$$

Thus $f \in \mathfrak{F}$ and so $\mathfrak{F}$ is non-empty.

Let B be a least upper bound for the derivatives $f'(z_0)$ as f ranges in $\mathfrak{F}$. Pick $g_i \in \mathfrak{F}$ such that $g_i'(z_0)$ approaches B. By (10.3) $\mathfrak{F}$ is a normal family. Therefore we may find a subsequence which converges to a holomorphic function $f(z)$. By (10.4) $f'(z_0) = B > 0$. Clearly $|f(z)| \leq 1$ on U, so that in fact $|f(z)| < 1$ on U by the open mapping theorem.

Now f is not constant as $f'(z_0) \neq 0$. Let $z_1 \in U$. Consider the functions $g_1(z) = g(z) - g(z_1)$, where $g \in \mathfrak{F}$. They are nowhere zero on $U - \{z_1\}$, as g is injective. As $f(z) - f(z_1)$ is a limit of such functions, and $f(z) - f(z_1)$ is not identically zero, (10.5) implies that $f(z) - f(z_1)$ is nowhere zero. But then f is injective. Thus $f \in \mathfrak{F}$ and $f(z)$ is an element of $\mathfrak{F}$ with maximal derivative at z_0.

Suppose that f is not surjective. Pick w_0 not in the image. As U is simply connected, we may find a holomorphic branch for

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}}.$$

Note that F is the composition of f, the automorphism of the unit disc

$$z \longrightarrow \frac{z - w_0}{1 - \bar{w}_0 z},$$

and the square root. Thus F is injective and $|F(z)| < 1$.

Let

$$G(z) = \frac{|F'(z_0)|}{F'(z_0)} \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} F(z)}.$$

Note that G is the composition of F and the automorphism of the unit disc

$$z \longrightarrow \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} z}.$$

Thus G is injective and $|G(z)| < 1$. Clearly $G(z_0) = 0$. Moreover $G'(z_0) > 0$. In fact

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{1 + |w_0|}{2\sqrt{|w_0|}} B > B,$$

a contradiction. □

The key point about the definition of G is as follows. We can use the formula for G to express f as a function of $w = G(z)$, which induces a map of the unit disc $|w| < 1$ to itself. The inequality

$$f'(z_0) < G'(z_0),$$

is then a consequence of Schwarz's lemma.

We now turn to the proof of (10.3). We will need to develop a lot of the theory of metric spaces and function spaces.