11. Normal families

Definition 11.1. *The functions in a family $\mathfrak{F}$ are said to be **equicontinuous** on a set $E \subset U$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f(z), f(z_0)) < \epsilon$ for all $|z - z_0| < \delta$, z, z_0 in E, simultaneously for all $f \in \mathfrak{F}$.*

Recall:

Definition 11.2. *A family $\mathfrak{F}$ is said to be **normal** on U if every sequence of functions f_n has a subsequence which converges uniformly on every compact subset of U.*

Definition 11.3. *We say that an increasing sequence of compacts subsets E_k **exhausts** U if*

$$U = \bigcup_k E_k.$$

Example 11.4. *It is easy to write down a sequence of subsets which exhausts U. For example, for each k let*

$$E_k = \{\, z \in U \mid |z| \leq k \text{ and } |z - z_0| \geq 1/k \text{ for all } z_0 \in \mathbb{C} - U \,\}.$$

We can make the set of functions from U to X into a metric space as follows. First replace the distance function d on X by

$$\delta(a, b) = \frac{d(a, b)}{1 + d(a, b)}.$$

It is easy to check that δ satisfies the triangle inequality and it is clear that δ is bounded. Given f and g, let

$$\delta_k(f, g) = \sup_{z \in E_k} \delta(f(z), g(z)).$$

Finally let

$$\rho(f, g) = \sum_{k=1}^{\infty} \delta_k(f, g) 2^{-k}.$$

Clearly ρ is finite and it is easy to check that with the definition of distance the space of all functions becomes a metric space.

Lemma 11.5. *A sequence of functions f_n converges uniformly to f on compact subsets if and only if it converges to f with respect to ρ.*

Proof. Suppose that f_n converges to f with respect to ρ. Pick a compact subset B and $\epsilon > 0$. Then we can find k such that $B \subset E_k$. By assumption we can find n_0 such that

$$\delta(f, f_n) \leq \frac{\epsilon}{2^k} \qquad \text{for all} \qquad n \geq n_0.$$

In this case

$$\delta_k(f, f_n) \leq \epsilon \qquad \text{for all} \qquad n \geq n_0,$$

so that

$$\delta(f(x), f_n(x)) \leq \epsilon \qquad \text{for all} \qquad n \geq n_0,\ x \in B.$$

Conversely suppose that f tends to f_n uniformly on compact subsets. Pick $\epsilon > 0$. Then we may find k_0 such that

$$\sum_{k=k_0}^{\infty} \frac{1}{2^k} < \frac{\epsilon}{2}.$$

Since f_n tends uniformly to f on E_{k_0}, we may find n_0 such that

$$\delta(f(x), f_n(x)) < \frac{\epsilon}{2} \qquad \text{for all} \qquad n \geq n_0,\ x \in E_{k_0}.$$

But then

$$\delta_k(f, f_n) \leq \frac{\epsilon}{2} \qquad \text{for all} \qquad k \leq k_0$$

so that

$$\begin{aligned} \delta(f, f_n) &= \sum_{k=1}^{k_0} 2^{-k}\delta_k(f, f_n) + \sum_{k>k_0} 2^{-k}\delta_k(f, f_n) \\ &\leq \epsilon. \end{aligned}$$

□

Recall that a metric space is compact if and only if every sequence has a convergent subsequence.

Theorem 11.6. *A family $\mathfrak{F}$ is normal if and only if its closure with respect to ρ is compact.*

Proof. We may suppose that $\mathfrak{F}$ is closed, and the result follows from (11.5). □

Recall some more notions from the theory of metric spaces:

Definition 11.7. *We say that a metric space X is* ***totally bounded*** *if for every $\epsilon > 0$ there are points $x_1, x_2, \dots, x_n \in X$ such that for every $x \in X$ we may find $1 \leq i \leq n$ such that $d(x, x_i) < \epsilon$.*

Lemma 11.8. *$Y \subset X$ is totally bounded if and only if its closure Z is totally bounded.*

Proof. One direction is clear; if Y is totally bounded then so is Z. Now suppose that Z is totally bounded. Pick $\epsilon > 0$. Then we may find $x_1, x_2, \dots, x_n \in Z$ such that for any $x \in Z$ we may find x_i such that $d(x, x_i) < \epsilon/2$. Pick $y_i \in Y$ such that $d(x_i, y_i) < \epsilon/2$. Then given $y \in Y$ we may find x_i such that $d(x_i, y) < \epsilon/2$. But then

$$d(y_i, y) \leq d(x_i, y_i) + d(x_i, y) < \epsilon.$$

□

Lemma 11.9. *If X is totally bounded then every sequence has a Cauchy subsequence.*

Proof. Let $y_1, y_2, \dots$ be an infinite sequence in X. We will inductively construct for each k a subsequence x_{nk} of x_{nk-1}. Given k we may find $x_1, x_2, \dots, x_m$ such that if $x \in X$ then $d(x, x_i) < 1/2k$. Therefore we may pick i such that

$$\{ j \mid d(y_j, x_i) < 1/2k \}$$

is infinite. Therefore we may find a subsequence x_{nk} such that $d(x_{nk}, x_i) < 1/2k$. It follows by the triangle inequality that $d(x_{nk}, x_{n'k}) < 1/k$. This finishes the construction of the subsequences.

The diagonal subsequence is the subsequence we are looking for. □

Lemma 11.10. *The set of functions from U to X is complete if and only if X is complete.*

Proof. Clear. □

Putting all of this together, we get:

Lemma 11.11. *If X is complete then $\mathfrak{F}$ is normal if and only if it is totally bounded.*

Proof. If X is complete then the space of functions from U to X is complete by (11.10). By (11.8) we may assume that $\mathfrak{F}$ is closed, so that $\mathfrak{F}$ is a complete totally bounded metric space. But then $\mathfrak{F}$ is compact by (11.9), so that it is normal by (11.6). □

We can restate some of this in terms of the space X, as opposed to $\mathfrak{F}$:

Proposition 11.12. *The family $\mathfrak{F}$ is totally bounded if and only if for every compact subset $E \subset U$ and every $\epsilon > 0$ it is possible to find $f_1, f_2, \dots, f_n \in \mathfrak{F}$ such that every $f \in \mathfrak{F}$ satisfies $\delta(f(x), f_j(x)) < \epsilon$, for some j, and every $x \in E$.*

Proof. Suppose that $\mathfrak{F}$ is totally bounded. Pick $\epsilon > 0$. By assumption we may find $f_1, f_2, \dots, f_n$ such that for every $f \in \mathfrak{F}$, we may find j such that $\rho(f, f_j) < \epsilon$.

Pick a compact subset E. Then we may find k such that $E \subset E_k$. Then we may find j such that $\delta(f(x), f_j(x)) < \frac{\epsilon}{2^k}$ for every $x \in E_k$. But then $\delta(f(x), f_j(x)) < \epsilon$ for every $x \in E \subset E_k$.

Now consider the reverse direction. Pick $\epsilon > 0$. Then we may find k_0 such that

$$2^{-k_0} < \frac{\epsilon}{2}.$$

Pick j such that

$$\delta(f(x), f_j(x)) < \frac{\epsilon}{2} \qquad \text{for every} \qquad x \in E_{k_0}.$$

Then

$$\begin{aligned}\rho(f, f_j) &= \sum_k \delta_k(f, f_j) \\ &= \sum_{k \leq k_0} \delta_k(f, f_j) + \sum_{k > k_0} \delta_k(f, f_j) \\ &< \epsilon.\end{aligned}$$

□

Theorem 11.13 (Ascoli-Arzola)**.** *A family of continuous functions $\mathfrak{F}$ with values in a complete metric space X is normal in the region $U \subset \mathbb{C}$ if and only if*

(1) $\mathfrak{F}$ is equicontinuous on every compact subset $E \subset U$, and
(2) for any $z \in U$ the values $f(z)$, $f \in \mathfrak{F}$ lie in a compact subset of X.

Proof. Suppose that $\mathfrak{F}$ is normal. We give two proofs that (1) must hold.

For the first proof, pick $\epsilon > 0$. By (11.12), we may find $f_1, f_2, \dots, f_n$ such that for every $f \in \mathfrak{F}$ we may find j such that $\delta(f(z), f_j(z)) < \frac{\epsilon}{3}$ for every $z \in E$. As each f_j is uniformly continuous on E, we can find $\delta > 0$ such that for every j and z, $z_0 \in E$ with $|z - z_0| < \delta$, $\delta(f_j(z), f_j(z_0)) < \frac{\epsilon}{3}$. But then

$$\delta(f(z), f(z_0)) \leq \delta(f(z), f_j(z)) + \delta(f_j(z), f_j(z_0)) + \delta(f_j(z_0), f(z_0)) < \epsilon.$$

For the second proof, suppose that $\mathfrak{F}$ is not equicontinuous on E. Then there is an $\epsilon > 0$, a sequence of pairs of points z_n, z'_n, with $|z_n - z'_n| \to 0$, and functions f_n such that $\delta(f_n(z), f_n(z')) \geq \epsilon$. As E is compact, possibly replacing z_n by a subsequence, we may assume that the points z_n and z'_n converge to a point $z \in E$. As $\mathfrak{F}$ is normal, possibly passing to a subsequence, we may also assume that the functions f_n converge to a function f uniformly on E. f is continuous on E, whence uniformly continuous on E.

Pick n_0 such that

$$\delta(f_n(z), f(z)) < \frac{\epsilon}{3} \qquad \text{and} \qquad \delta(f(z_n), f(z'_n)) < \frac{\epsilon}{3},$$

for all $z \in E$ and $n \geq n_0$. Then

$$\delta(f_n(z_n), f_n(z'_n)) \leq \delta(f_n(z_n), f(z_n)) + \delta(f(z_n), f(z'_n)) + \delta(f(z'_n), f_n(z'_n)) < \epsilon,$$

a contradiction.

We now turn to the proof of (2). Pick $z \in U$. We will show that the closure of the set

$$\{\, f(z) \mid f \in \mathfrak{F} \,\},$$

is compact. Let w_n be a sequence in the closure. For each n, pick $f_n \in \mathfrak{F}$ such that $\delta(f_n(z), w_n) < 1/n$. As $\mathfrak{F}$ is a normal family, passing to a subsequence, we may assume that f_n converges to a continuous function f. But then w_n converges to $f(z)$.

Now suppose that (1) and (2) holds. We want to show that $\mathfrak{F}$ is normal. Enumerate ζ_n the points with rational coordinates. Then the points ζ_n are everywhere dense. Let f_n be a sequence of functions in $\mathfrak{F}$. We are going to iteratively construct subsequences n_{ik} for each k. n_{ik} will be a subsequence of n_{ik-1} such that $f_{n_{ik}}(\zeta_k)$ converges. The existence of these subsequences is immediate from (2). Let f_n be the sequence obtained by taking the diagonal sequence. Then the sequence $f_n(\zeta_k)$ converges for all k.

Suppose that $E \subset U$ is a compact subset. We will show that f_n converges uniformly on E. Pick $\epsilon > 0$. As $\mathfrak{F}$ is equicontinuous we may find $\delta > 0$ such that

$$\delta(f(z), f(z')) < \frac{\epsilon}{3} \qquad \text{whenever} \qquad |z - z'| < \delta,\ f \in \mathfrak{F}.$$

As E is compact, it is covered by finitely many balls of radius $\delta/2$. Pick one ζ_k from each such ball. Then we may find an index n_0 such that

$$\delta(f_n(\zeta_k), f_m(\zeta_k)) < \frac{\epsilon}{3} \qquad \text{for all} \qquad n, m \geq n_0.$$

Pick $z \in E$. Then $|z - \zeta_k| < \delta$ for at least one k. But then

$$\begin{aligned}\delta(f_n(z), f_m(z)) &\leq \delta(f_n(z), f_n(\zeta_k)) + \delta(f_n(\zeta_k), f_m(\zeta_k)) + \delta(f_m(\zeta_k), f_m(z)) \\ &< \epsilon.\end{aligned}$$

As the values $f_n(z)$ belong to a compact subset, and X is complete, the pointwise limit f of the f_n exists, and f is a uniform limit of the f_n on E. □