12. Normality and holomorphic functions

Finally we are able to prove

Theorem 12.1. *A family $\mathfrak{F}$ of holomorphic functions is normal if and only if the functions are uniformly bounded on compact subsets.*

Proof. Since a subset of $\mathbb{C}$ is contained in a compact subset if and only if it is bounded, by Ascoli-Arzola it suffices to prove that $\mathfrak{F}$ is equicontinuous.

Let γ be the boundary of a closed ball of radius r contained in U. Pick z and z_0 in the interior of this ball. Then Cauchy's formula implies

$$\begin{aligned} f(z) - f(z_0) &= \frac{1}{2\pi i}\int_\gamma \left(\frac{1}{w-z} - \frac{1}{w-z_0}\right) f(w)\,dw \\ &= \frac{z-z_0}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z)(w-z_0)}. \end{aligned}$$

If $|f| \leq M$ on γ and z and z_0 belong to the ball of radius $r/2$ with the same centre, then

$$|f(z) - f(z_0)| < \frac{4M|z-z_0|}{r}.$$

Let E be a compact subset of U. For each point of E we may find a ball of (variable) radius r centred about this point contained in U. Since balls of radius $r/4$ about these points forms an open cover of E and E is compact, we may cover E by finitely many balls of radius $r/4$. Let r_k, a_k and M_k be the radius, centre and supremum of $|f|$ on the kth circle, for $f \in \mathfrak{F}$. Let r be the minimum of the r_k and let M be the maximum of the M_k. Given $\epsilon > 0$ let

$$\delta = \min(\frac{r}{4}, \frac{\epsilon r}{4M}).$$

Suppose that $|z - z_0| < \delta$. Then we may find a_k such that $|z_0 - a_k| < \frac{r_k}{4}$. Then

$$|z - a_k| \leq |z - z_0| + |z_0 - a_k| < \delta + \frac{r_k}{4} \leq \frac{r_k}{2}.$$

But then

$$|f(z) - f(z_0)| < \frac{4M_k|z-z_0|}{r_k} < \frac{4M\delta}{r} \leq \epsilon. \qquad \square$$

Remark 12.2. *Note that to say that a function is uniformly bounded on compact subsets is equivalent to the statement that it is locally bounded.*

Theorem 12.3. *A locally bounded family of functions has locally bounded derivatives.*

Proof. If γ is the boundary of a closed disk in U of radius r then

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(z-w)^2}.$$

Thus

$$|f'(z)| \leq \frac{4M}{r}$$

in the concentric disc of radius at most $r/2$, where M is the maximum value of $|f(z)|$ on γ. □

It turns out that one can extend the notion of normality of holomorphic functions, if one considers the image to be the whole extended complex plane, $\mathbb{P}^1$. In this case one might as well allow meromorphic functions.

Definition 12.4. *A family of holomorphic functions in a region U is* ***normal*** *if every sequence has a subsequence that converges uniformly on every compact subset $E \subset U$ either to a function f, or to ∞.*

Note that (12.4) is the correct notion of normality for the Riemann sphere with the standard Euclidean metric. To see this, we will need to extend the Weierstrass and the Hurwitz theorem to this setting:

Lemma 12.5. *A sequence of meromorphic (respectively holomorphic) functions converges in the standard metric on the Riemann sphere if and only if the function converges, uniformly on every compact subset, to a meromorphic (respectively holomorphic) function or to ∞.*

Proof. Suppose that $f_1, f_2, \dots$ converges to f in the standard metric. Then f is continuous. If $f(z_0) \neq \infty$ then by continuity f_n are bounded in a neighbourhood of z_0 and by the ordinary Weierstrass theorem f is holomorphic in a neighbourhood of z_0. If $f(z_0) = \infty$ then $1/f$ is the limit of $1/f_n$. Thus $1/f$ is holomorphic near z_0 and so f is meromorphic.

If in addition $f_n(z)$ are holomorphic then Hurwitz implies that $1/f$ is identically zero so that $f = \infty$. □

Note that the derivatives of a normal family are not necessarily a normal family. For example consider

$$f_n(z) = n(z^2 - n).$$

These functions tend to infinity, uniformly on compact subsets. But the derivatives

$$f_n(z) = 2nz$$

don't tend to infinity everywhere. If $z \neq 0$ the limit is still infinite but if $z = 0$ the limit is zero.