14. Subharmonic functions

Definition 14.1. Let \( u: U \rightarrow \mathbb{R} \) be a continuous function. We say that \( u \) satisfies the **mean-value property** if
\[
u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta,
\]
whenever the disk \( |z - z_0| \leq r \) is contained in \( U \).

Note that the mean-value property implies the maximum principle. In fact it suffices that the mean-value property holds in a small ball about every point of \( z_0 \in U \), whose radius depends on \( z_0 \).

Theorem 14.2. A continuous function \( u(z) \) on a domain \( U \) satisfies the mean-value property if and only if it is harmonic.

**Proof.** If \( u \) is harmonic we have already seen that it must satisfy the mean-value property.

Now suppose that \( u \) satisfies the mean-value property. Let \( v \) be any harmonic function. Then the difference \( u - v \) also satisfies the mean-value property. In particular the difference satisfies the maximum principle.

Pick a ball contained in \( U \) for which \( u \) satisfies the mean-value property. Let \( U \) be the restriction of \( u \) to the boundary of this ball and let \( v = P_U \) be the Poisson integral. Then \( v \) is a harmonic function which agrees with \( u \) on the boundary of the ball. Thus the difference \( u - v \) satisfies the mean-value property and is zero on the boundary. By the maximum principle \( u - v \) is zero so that \( u = v \) is harmonic on the ball. Thus \( u \) is harmonic. \( \square \)

Using (14.2) it is clear that we could have defined a function to be harmonic if and only if it satisfies the mean-value property, so that we can give a definition of harmonic which makes no reference to the existence of partial derivatives.

Theorem 14.3 (Harnack’s inequality). Let \( u \) be a harmonic function on a disc centred at the origin of radius \( \rho \) and let \( z \) be a point of distance \( r \) away from the origin.

If \( u(z) \geq 0 \) then
\[
\frac{\rho - r}{\rho + r} u(0) \leq u(z) \leq \frac{\rho + r}{\rho - r} u(0).
\]

**Proof.** Note that
\[
\frac{\rho - r}{\rho + r} \leq \frac{\rho^2 - r^2}{|pe^{i\theta} - z|^2} \leq \frac{\rho + r}{\rho - r}.
\]
Thus
\[ |u(z)| \leq \frac{1}{2\pi} \frac{\rho + r}{\rho - r} \int_0^{2\pi} |u(\rho e^{i\theta})| \, d\theta. \]
If further \( u(\rho e^{i\theta}) \geq 0 \) then we can apply the first inequality as well to obtain
\[ \frac{1}{2\pi} \frac{\rho - r}{\rho + r} \int_0^{2\pi} u \, d\theta \leq u(z) \leq \frac{1}{2\pi} \frac{\rho + r}{\rho - r} \int_0^{2\pi} u \, d\theta. \]
Now use the fact that the integral is equal to the value of \( u \) at the origin. \( \square \)

**Theorem 14.4** (Harnack’s principle). Let \( u_n(z) \) be a harmonic function on a region \( U_n \). Suppose that \( U \) is a region such that every point of \( U \) has a neighbourhood which is contained in all but finitely many \( U_n \) and that \( u_n(z) \leq u_{n+1}(z) \) for all but finitely many \( n \) on the same neighbourhood.

Then either \( u_n(z) \) tends uniformly to \( \infty \) on compact subsets or it tends to a harmonic function \( u(z) \), uniformly on compact subsets.

**Proof.** Suppose first that there is a \( z_0 \in U \) such that \( u_n(z_0) \) tends to \( \infty \). Then we may find \( r > 0 \) and \( m \) such that the functions \( u_n, n > m \) are non-decreasing for \( |z - z_0| < r \). Applying the left side of Harnack’s inequality to \( u = u_n - u_m \), we see that \( u_n \) tends uniformly to \( \infty \) in the disk \( |z - z_0| \leq r/2 \). Similarly if the limit at \( z_0 \) is finite, then the right side of Harnack’s inequality implies that \( u_n \) converges uniformly to a finite number on the disk \( |z - z_0| \leq r/2 \).

Thus the set of points where \( u_n \) tends either to zero or to infinity are both open. As \( U \) is connected one such set must be empty. If the limit is infinite, uniformity follows by the usual compactness argument.

Now suppose that the limit is finite everywhere. Then
\[ u_{n+p}(z) - u_n(z) \leq 3(u_{n+p}(z_0) - u_n(z_0)) \quad \text{for} \quad |z - z_0| \leq r/2, \]
and \( n + p \geq n \geq m \). Thus convergence at \( z_0 \) implies uniform convergence in a neighbourhood of \( z_0 \). Thus we get that \( u_n \) tends to a limit \( u \) uniformly on compact subsets. But \( u(z) \) must be harmonic by Poisson’s formula. \( \square \)

Note that Laplace’s equation in one variable reduces to requiring that the second derivative is zero. In this case the solutions to Laplace’s equation are the linear functions. A function on an interval, which is less than the linear function with the same values at the endpoints of the interval, is called convex. A subharmonic function is any function of two variables with the same property. Any function which is less
than a harmonic function with the same boundary values is called sub-
harmonic. Since this definition is a little unmanageable, we are lead
to:

**Definition 14.5.** A continuous function \( v(z) \) on a region \( U \) is said to be **subharmonic** in \( U \) if for any harmonic function \( u \) defined on any open subset \( U' \subset U \) the difference \( v - u \) satisfies the maximum principle in \( U' \) (that is, if \( v - u \) has a maximum then it is constant).

Note that the condition that \( v \) is subharmonic is local in nature. In fact if we say that \( v \) is subharmonic at \( z_0 \) if it is subharmonic in a neighbourhoud of \( z_0 \), then \( v \) is subharmonic if and only if it is subharmonic at each point of \( U \).

Note that a harmonic function is subharmonic. Note that if \( v \) is \( C^2 \) and \( z_0 \) is a maximum of \( v - u \) then the partials of \( v - u \) at \( z_0 \) vanish and the 2nd derivative is non-positive, so that \( \Delta v = \Delta (v - u) \leq 0 \). In fact this is enough to characterise subharmonic functions whose second partials exist:

**Lemma 14.6.** Let \( v \) be a continuous function whose second partial derivatives exist on \( U \).

Then \( v \) is subharmonic if and only if \( \Delta v \geq 0 \) on \( U \).

*Proof.* Exercise left to the reader. \( \square \)

Note however that there are many subharmonic functions whose derivatives do not exist.

**Theorem 14.7.** A continuous function \( v(z) \) is subharmonic if and only if it satisfies the inequality

\[
v(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta}) \, d\theta,
\]

for every disk \( |z - z_0| \leq r \) contained in \( U \).

*Proof.* Suppose that \( v \) satisfies the inequality. Let \( u \) be a harmonic function. Suppose that \( z_0 \) is a maximum of \( v - u \). Then \( v - u \) also satisfies the inequality. Replacing \( v \) by \( v - u \), we may suppose that \( z_0 \) is a maximum of \( v \). Suppose that the disk \( |z - z_0| < r \) is contained in \( U \). If \( v(z) < v(z_0) \) then the integral on the right must be less than \( v(z_0) \), since it is an average, a contradiction. Thus \( v \) is constant on the boundary of the disk, whence it is constant on the disc, by varying the radius \( r \). But then \( v \) must be constant, so that \( v \) is subharmonic.

Now suppose that \( v \) is subharmonic. Let \( V \) be the restriction of \( v \) to the boundary of the disk \( |z - z_0| < r \) and let \( P_V \) be the corresponding Poisson integral. Then \( P_V \) is harmonic so that \( v - P_V \) satisfies the
maximum principle on the disk. It is zero on the boundary of the disk, so that
\[ v \leq P_v \]
in the interior. □

Here are some basic properties of subharmonic functions:

**Lemma 14.8.** Suppose that \( v = v_1 \) and \( v_2 \) are subharmonic. Then so is

1. \( kv \), for any constant \( k \geq 0 \),
2. \( v_1 + v_2 \),
3. \( w = \max(v_2, v_2) \), and
4. the function \( v' \) which is equal to \( v \) outside a disc and \( P_v \) inside a disc.

**Proof.** (1) and (2) follow from (14.7).

Let \( u \) be any harmonic function such that \( w - u \) has a maximum at \( z_0 \). Replacing \( v_i \) by \( v_i - u \) we may assume that \( u = 0 \). Suppose that \( w(z_0) = v_1(z_0) \). Then
\[ v_1(z) \leq w(z) \leq w(z_0) = v_1(z_0). \]
Thus \( v_1(z) \) is constant. Now either \( w(z) = v_1(z) \) in a neighbourhood of \( z_0 \) or \( w(z_0) = v_2(z_0) \). But then \( v_2(z) \) is also constant in a neighbourhood of \( z_0 \) and either way it follows that \( w(z) \) is constant in a neighbourhood of \( z_0 \). But then \( w(z) \) satisfies the maximum principle. Thus \( w \) is subharmonic. This is (3).

Note that \( v' \) is continuous since \( P_v \) is equal to \( v \) on the boundary of the disk. On the other hand as \( P_v \) is a harmonic function, it is certainly subharmonic. We can check that \( v' \) is subharmonic at every point \( z_0 \). Suppose that \( z_0 \) is a maximum of \( v' \). If \( z_0 \) belongs to the interior of the disc we are done since \( P_v \) is subharmonic. If \( z_0 \) belongs to the boundary of the disc then \( z_0 \) is a local maximum of \( v \). But then \( v \) is constant and so \( P_v \) is constant. Thus is (4). □