15. The Dirichlet Problem: Perrons method

Let U be a bounded region and let $f\colon \Gamma \longrightarrow \mathbb{R}$ be a continuous function defined on the boundary Γ of U. The Dirichlet problem is to determine a harmonic function u which is equal to f on the boundary.

To describe Perron's method it is not even necessary to assume that $f(\zeta)$ is continuous; for simplicity we will however assume that there is a constant M such that $|f(\zeta)| \leq M$ (for clarity we will use the variable ζ to denote values on the boundary Γ).

To each such function f, Perron's method associates a harmonic function u on U; whenever f is continuous and U satisfies some reasonable conditions then u will extend to a continuous function on the closure of U which agrees with f on Γ.

Definition 15.1. *The **Perron family** associated to f is*

$$\mathcal{P}(f) = \{\, v\colon U \longrightarrow \mathbb{R} \,|\, v \text{ is subharmonic and } \limsup_{z\to\zeta} v(z) \leq f(\zeta) \,\},$$

where ζ ranges over the whole of Γ.

Here the use of limsup means precisely that given any point $\zeta \in \Gamma$ and any $\epsilon > 0$ there is a disk Δ of radius δ about ζ such that if $z \in \Delta \cap U$ then

$$v(z) < f(\zeta) + \epsilon.$$

Definition 15.2. *The **Perron function** u associated to f is the function*

$$u(z) = \sup\{\, v(z) \,|\, v \in \mathcal{P}(f) \,\}.$$

Lemma 15.3. *The Perron function u associated to f is harmonic.*

Proof. We first prove that for any function $v \in \mathcal{P}(f)$, we have $v \leq M$. Even though this follows from the maximum principle in a fairly straightforward fashion, we will go through the proof of this in detail, since it is quite important.

Given $\epsilon > 0$, let

$$E = E_\epsilon = \{\, z \in U \,|\, v(z) \geq M + \epsilon \,\}.$$

The points in $\mathbb{C} - E$ are of three kinds

(1) points in the exterior of U,
(2) points on Γ and
(3) points in U with $v(z) < M + \epsilon$.

For points of type (1) we may find a small disk containing the point completely contained in the exterior of U. In case (2) we may find a neighbourhood Δ of the point such that $v(z) < M+\epsilon$ for $z \in \Delta \cap U$. In case (3) by continuity there is a neighbourhood in U such $v(z) < M+\epsilon$. It follows that the complement of E is open, so that E itself is closed. Moreover as U is bounded, E is compact. Suppose that E is non-empty. Thus v achieves its maximum on E and so v is constant and greater than $M+\epsilon$, which contradicts the fact that $v \in \mathcal{P}(f)$. Thus E is empty and $v \leq M$ on U.

Let Δ be a disk whose closure is contained in U. Let $z_0 \in \Delta$. Then we may find a sequence of functions $v_1, v_2, \ldots \in \mathcal{P}(f)$ such that

$$u(z_0) = \lim_{n\to\infty} v_n(z_0).$$

Let

$$V_n = \max(v_1, v_2, \ldots, v_n).$$

Then the functions V_n are a non-decreasing sequence of functions in $\mathcal{P}(f)$, since the maximum of a finite set of subharmonic functions is subharmonic, and the correct behaviour at the boundary is clear. Let V'_n be the subharmonic function which is equal to V_n outside Δ and which is harmonic inside Δ. Then $V'_n \in \mathcal{P}(f)$ and the sequence of functions V'_n is also non-decreasing. Moreover the sequence of inequalities

$$v_n(z_0) \leq V_n(z_0) \leq V'_n(z_0) \leq u(z_0),$$

shows that

$$\lim_{n\to\infty} V'_n(z_0) = u(z_0).$$

By Harnack's principle, the sequence of functions $V'_1, V'_2, \ldots$ converges to a harmonic function U on Δ, for which $U \leq u$ and $U(z_0) = u(z_0)$.

Now pick another point $z_1 \in \Delta$. We go through the same construction as before. Pick $w_1, w_2, \ldots \in \mathcal{P}(f)$ so that

$$u(z_1) = \lim_{n\to\infty} w_n(z_1).$$

But now we put in an added twist and replace w_n by $\max(v_n, w_n)$. Repeating the construction we obtain a harmonic function $U \leq U_1 \leq u$ on Δ such that $U_1(z_1) = u(z_1)$. Now the harmonic function $U - U_1$ has a maximum at z_0, namely zero. But then $U = U_1$ so that $U(z_1) = u(z_1)$. As z_1 is arbitrary, $u = U$ on Δ and so u is harmonic in Δ. But then u is harmonic everywhere, since Δ is arbitrary. □

Now we investigate the circumstances under which u is a solution of the Dirichlet problem. First note that the Dirichlet problem does not always have a solution. For example let U be the punctured unit disk $0 < |z| < 1$ and let f be the function which is zero on the boundary and 1 at the origin. A harmonic function with these boundary values would be bounded. In particular it would have a removable singularity at the origin. But then the maximum principle would imply that the function is identically zero. In particular it would not have the correct behaviour at the origin.

On the other hand, suppose that U is a solution. Then $U \in \mathcal{P}(f)$. Hence $U \leq u$. The opposite inequality $u \leq U$ follows by the maximum principle. Thus a solution to the Dirichlet problem, if it exists at all, must equal u.

In fact a solution to the Dirichlet problem always exists if U is not "too thin" at any boundary point. We can measure this in a fashion which at first seems to have nothing to do with this notion.

Lemma 15.4. *Suppose that ω is a harmonic function on U which extends to a continuous function on the boundary where it is strictly positive except at one point ζ_0, where it is zero (any such function is called a **barrier**).*

If f is continuous at ζ_0 then

$$\lim_{z \to \zeta_0} u(z) = f(\zeta_0).$$

Proof. It surely suffices to prove that for every $\epsilon > 0$

$$\limsup_{z \to \zeta_0} u(z) \leq f(\zeta_0) + \epsilon \qquad \text{and} \qquad \liminf_{z \to \zeta_0} u(z) \geq f(\zeta_0) - \epsilon.$$

Pick a neighbourhood Δ of ζ_0 such that

$$|f(\zeta) - f(\zeta_0)| < \epsilon,$$

for every $\zeta \in \Delta$. On $\bar{U} - U \cap \Delta$ the function ω has a positive minimum ω_0. Consider the boundary values of the harmonic function

$$W(z) = f(\zeta_0) + \epsilon + \frac{\omega(z)}{\omega_0}(M - f(\zeta_0)).$$

For $\zeta \in \Delta$ we have

$$W(\zeta) \geq f(\zeta_0) + \epsilon > f(\zeta),$$

and for ζ outside Δ we have

$$W(\zeta) \geq M + \epsilon > f(\zeta).$$

By the maximum principle any function $v \in \mathcal{P}(f)$ must satisfy

$$v(z) < W(z).$$

In particular

$$u(z) < W(z).$$

But then

$$\limsup_{z \to \zeta_0} u(z) \leq W(\zeta_0) = f(\zeta_0) + \epsilon.$$

To prove the other inequality it suffices to prove that the harmonic function

$$V(z) = f(\zeta_0) - \epsilon - \frac{\omega(z)}{\omega_0}(M - f(\zeta_0)),$$

is an element of $\mathcal{P}(f)$. For $\zeta \in \Delta$ we have

$$V(\zeta) \leq f(\zeta_0) - \epsilon < f(\zeta),$$

and for any other boundary point,

$$V(\zeta) \leq -M - \epsilon < f(\zeta).$$

Thus $V \in \mathcal{P}(f)$ and so $V(z) \leq U(z)$. But then

$$\liminf_{z \to \zeta_0} u(z) \geq V(\zeta_0) = f(\zeta_0) - \epsilon,$$

which is the other inequality. □

Clearly this gives a sufficient way to prove that the Dirichlet problem has a solution; show that there is a barrier at every point. It remains to come up with a nice criteria for the existence of a barrier.

To warm up, suppose that a point $\zeta_0 \in U \cup \Gamma$ has a supporting hyperplane. That is to say that every point of $U \cup \Gamma$ is on one side of a half-plane except the point ζ_0, which lies on the boundary line. If the direction of the line is given by α, with the half-plane to the left, then

$$\omega(z) = \operatorname{Im}(e^{-i\alpha}(z - \zeta_0)),$$

is a barrier at ζ_0.

More generally suppose that ζ_0 is the endpoint of a line segment, whose other points are exterior to U. Let ζ_1 be the other endpoint. Pick a holomorphic branch of the function

$$\sqrt{\frac{z - \zeta_0}{z - \zeta_1}}.$$

For an appropriate choice of angle α it follows that

$$\operatorname{Im}\left[e^{-i\alpha}\sqrt{\frac{z - \zeta_0}{z - \zeta_1}}\right],$$

is a barrier at ζ_0.

Although this is not the best result possible, it is certainly quite strong:

Theorem 15.5. *The Dirichlet problem can be solved for any region U such that each boundary point is the end point of a line segment whose other points are exterior to U.*