17. Maps between Riemann surfaces: I

Definition 17.1. *We say that $f \colon X \longrightarrow Y$ is a **ramified cover** of surfaces if X and Y are surfaces and there is a discrete set of points in X, whose images in Y are also discrete, such that the removal of these points in X and Y makes f an unramified cover. The set of points in X where f is not is not locally an unramified cover is called the set of **ramification points**. The image of the ramification points in Y is called the set of **branch points**.*

Note that a ramified cover is not a local homeomorphism about any of the ramification points. Note also that the inverse image of a branch point need not be a ramification point.

Example 17.2. *Let $f \colon X = \mathbb{C} \longrightarrow Y = \mathbb{C}$ be the map $z \longrightarrow z^2$. Then f is a ramified cover. The set of ramification points is equal to the set $\{0\} \subset X$ and the set of branch points is equal to the set $\{0\} \subset Y$. The same is true for any of the maps $z \longrightarrow z^n$, $n \geq 1$. The map $f \colon \mathbb{C} \longrightarrow \mathbb{C}^*$ given by $z \longrightarrow e^z$ is also an unramified cover. In fact f is the universal cover of $\mathbb{C}^*$.*

Definition 17.3. *Let $f \colon X \longrightarrow Y$ be a map of topological spaces. We say that f is **closed** (respectively **open**) if the image of every closed (respectively open) subset is closed (respectively open).*

Lemma 17.4. *Suppose that $f \colon X \longrightarrow Y$ is a continuous map of Hausdorff topological spaces and that X is compact.*

Then f is closed.

Proof. As X is compact Hausdorff $F \subset X$ is closed if and only if it is compact. But the image of every compact subset is compact and a compact subset of a Hausdorff space is closed. □

Lemma 17.5. *Let $f \colon X \longrightarrow Y$ be a continuous and closed map of metric spaces, which is a local homeomorphism.*

If $x_1, x_2, \dots$ is a Cauchy sequence of points of X whose images $y_1, y_2, \dots$ converge to a point $y \in Y$ then $x_1, x_2, \dots$ converge to a point $x \in X$ such that $f(x) = y$.

Proof. If these points do not have a limit, then the set

$$\{ x_i \,|\, i \in \mathbb{N} \},$$

is closed. But then its image

$$\{ y_i \,|\, i \in \mathbb{N} \},$$

is closed, which contradicts the fact that y is a point of the closure. □

Theorem 17.6. *Let $f\colon X \longrightarrow Y$ be a continuous map of topological surfaces. Suppose that*

- *f is closed, and*
- *there is a discrete set of points in X, such that f is a local homeomorphism outside this set.*

Then f is a ramified cover.

Proof. As f is closed the image of a discrete set of points is discrete. Since this result is local about Y, we may assume that Y is the open unit ball in $\mathbb{C}$, and that if f is not a local homeomorphism then the inverse image of 0 is one point, which is the unique point where f is not a local homeomorphism. Let $X' = X$ and $Y' = Y$ if f is there is no point where f is not a local homeomorphism; otherwise let $X' = X - \{f^{-1}(0)\}$ and $Y' = Y - \{0\}$.

For the first statement, it suffices to prove that $g = f|_{X'}\colon X' \longrightarrow Y'$ is an unramified cover. By (16.9), it suffices to prove that g satisfies the lifting property. Let $\gamma\colon [0,1] \longrightarrow Y'$ be a continuous map and let $x \in X'$ be a point such that $f(x) = y = \gamma(0)$. Let

$$E = \{\, s \in [0,1] \,|\, \text{we may lift } \gamma|_{[0,s]} \text{ to } \psi\colon [0,s] \longrightarrow X, \text{ with } \psi(0) = x \,\}.$$

Note that E is non-empty, since it contains 0, and that E is an interval. As in the proof of (16.7), E is open. Suppose that $E = [0,s)$ is not closed. Pick a sequence of real numbers $s_1, s_2, \dots$ tending to s such that we may lift $\gamma|_{[0,s_i]}$ to $\psi_i\colon [0,s_i] \longrightarrow X$. Let $x_i = \psi_i(s_i)$ and $y' = \gamma(s)$. Then the points $y_i = f(x_i)$ converge to y', so that by (17.5) the points x_i converge to x' a point of X such that $f(x') = y'$. But then we may lift $\gamma|_{[0,s]}$ to $\psi\colon [0,s] \longrightarrow X$ by defining

$$\psi(t) = \begin{cases} \psi_i(t) & \text{if } t < s_i \\ x & \text{if } t = s, \end{cases}$$

a contradiction. Thus $E = [0,1]$. □

Corollary 17.7. *Let $f\colon X \longrightarrow Y$ be a holomorphic map between Riemann surfaces, which is not locally constant.*

If f is closed then f is a ramified cover. In particular if X is a compact connected Riemann surface then every non-constant holomorphic map from X to any other Riemann surface is a ramified cover.

Proof. Note that given a point $x \in X$ it makes sense to talk about the derivative of f being zero at x. Indeed just pick a coordinate chart and consider the derivative between the corresponding open subsets of $\mathbb{C}$; the key point is that this is independent of the chart we pick, even if the value of the derivative is not.

Let $x \in X$ be a point of f where the derivative of f is not zero. Then f is locally injective and so f is a local homeomorphism about $x \in X$ by the open mapping theorem (indeed these statements are all local, in which case we are reduced to the classical case when X and Y are open subsets of $\mathbb{C}$).

On the other hand, the set of points where the derivative of f is zero is discrete, since this is true for any non-zero holomorphic map between open subsets of $\mathbb{C}$. Thus the result follows by (17.6). □

The really interesting fact is that the converse result is also true:

Theorem 17.8. *Let $f\colon X \longrightarrow Y$ be a ramified cover.*

If Y is a Riemann surface, then there is a unique choice of a Riemann surface structure on X such that f becomes holomorphic.

(17.8) is a special case of Riemann's existence theorem. We only sketch the proof. Define an altas on X by taking all charts $h_1\colon U \longrightarrow U' \subset \mathbb{C}$ such that the map $h_2 \circ f \circ h_1^{-1}\colon U'' \longrightarrow V'$ is holomorphic, where $h_2\colon V \longrightarrow V' \subset \mathbb{C}$ is any chart on Y and $U'' = (f \circ h^{-1})^{-1}(V) \subset U$. As long as this does give a holomorphic structure, uniqueness is clear since this atlas is maximal.

The key point is to check what happens at ramification points. Topologically we have a map between unit discs, which is an unramified cover of the punctured disc. The set of such maps is classified by the map on fundamental groups,

$$\mathbb{Z} \simeq \pi_1(X - x, x_0) \longrightarrow \pi_1(Y - y, y_0) \simeq \mathbb{Z}.$$

The only invariant of this map is the cokernel, which is a finite cyclic group,

$$\frac{\mathbb{Z}}{n\mathbb{Z}},$$

for a unique positive integer n. But the map $z \longrightarrow z^n$ has exactly this cokernel and this map is holomorphic. The number n is called the **ramification index**.

Note that this gives a way to construct very many Riemann surfaces X, since it is easy to write down lots of ramified covers between a compact oriented surface of genus g and $\mathbb{P}^1$.