18. Maps between Riemann surfaces: II

Note that there is one further way we can reverse all of this. Suppose that X instead of Y is a Riemann surface. Can we put a Riemann surface structure on Y such that f is holomorphic? Well there is one trivially necessary condition.

Definition 18.1. *Let $f\colon X \longrightarrow Y$ be an unramified cover of topological spaces or a ramified cover of surfaces. A* ***deck transformation*** *$g\colon X \longrightarrow X$ is any continuous map over Y, that is, such that $f = f \circ g$.*

Note that the set of all deck transformations forms a group, called the **group of deck transformations**.

Note that any map over Y permutes the fibres of f. Thus a deck transformation permutes, at least locally, the decks of f. Now if f is a holomorphic map of Riemann surfaces, then any deck transformation is holomorphic. Thus a trivially necessary condition to put a structure of Riemann surface on Y is that any deck transformation is holomorphic. This trivially necessary condition is in fact sufficient in a large number of examples.

Definition 18.2. *Let X be a topological space and let $G \subset \mathrm{Aut}(X)$ be a subgroup of the automorphism group. We say that the action of G on X is* ***properly discontinuous*** *if for every two compact subsets A and B of X the number of automorphisms $\phi \in G$ such that $\phi(A)$ intersects B is non-empty is finite.*

Theorem 18.3. *Let X be a Riemann surface. Let $G \subset \mathrm{Aut}(X)$ be a subgroup of the automorphism group of X, whose action on X is properly discontinuous.*

Then the quotient topological space $f\colon X \longrightarrow Y = X/G$ is naturally a Riemann surface, in such a way that f is holomorphic.

We prove this in a sequence of steps:

Lemma 18.4. *Let X be a topological space, let G be a subgroup of the group of automorphisms of X and let $f\colon X \longrightarrow Y = X/G$ the quotient of X modulo G.*

Then f is open.

Proof. Pick $U \in X$ and let $V = f(U)$. We have to show that $f^{-1}(V)$ is open. But $\phi(U)$ is open for any $\phi \in G$ and

$$f^{-1}(V) = \bigcup_{\phi \in G} \phi(U). \qquad \square$$

Lemma 18.5. *Let X be a locally compact Hausdorff topological space and let $G \subset \operatorname{Aut}(X)$ be a subgroup of the automorphism group, whose action on X is properly discontinuous.*

Then the quotient $f\colon X \longrightarrow Y = X/G$ is Hausdorff and f is closed.

Proof. Let $y_1 \neq y_2$ be two distinct points of Y. Pick $x_i \in X$ such that $f(x_i) = y_i$, $i = 1$ and 2. Pick a compact neighbourhood of x_2. Then there are only finitely many elements of G such that $\phi(x_1)$ is in this neighbourhood. So we may find a compact neighbourhood A_2 of x_2 such that $\phi(x_1) \notin A_2$ for any $\phi \in G$. Similarly we may then find a compact neighbourhood A_1 of x_1 such that $\phi(A_1) \cap A_2$ is always empty. But then $f(A_1)$ and $f(A_2)$ are disjoint neighbourhoods of y_1 and y_2.

Now let $F \subset X$ be a closed subset and let $H = f(F)$. We have to show that $f^{-1}(H)$ is closed. Now $\phi(F)$ is closed for all $\phi \in G$ and

$$f^{-1}(H) = \bigcup_{\phi \in G} \phi(F).$$

Since the action is properly disontinuous, this union is closed. □

Lemma 18.6. *Let X be a manifold and let $G \subset \operatorname{Aut}(X)$ be a subgroup of the automorphism group, whose action is properly discontinuous such that if we can find $\phi \in G$ and $x \in X$ with $\phi(x) = x$ then ϕ is the identity.*

Then $f\colon X \longrightarrow Y = X/G$ is an unramified cover of manifolds.

Proof. As in the proof of (18.5), it is clear that f is a local homeomorphism. Since f is closed as well, it is easy to check that the lifting property holds. But then f is an unramified cover. □

Lemma 18.7. *Let X be a surface and let $G \subset \operatorname{Aut}(X)$ be a subgroup of the automorphism group, whose action is properly discontinuous.*

Then $f\colon X \longrightarrow Y = X/G$ is a ramified cover of surfaces.

Proof. Let R be the set of points $x \in X$ where there is an element $\phi \in G$, not equal to the identity, such that $\phi(x) = x$. Since the action of G is properly discontinuous R is a discrete subset, whose image $B = f(R)$ is also discrete.

Let $U = X - R$ and $V = Y - B$. Then G acts on U and the quotient is V, so that the result follows by (18.6). □

Proof of (18.3). By (18.7) we have a ramified cover of topological surfaces. Pick an open cover of X, which has the property that every ramification point is only contained in one open subset. Then we may choose a coordinate chart about every branch point such that the map is locally given by $z \longrightarrow z^n$ about a ramification point.

Now suppose that $y \in Y$ is not a branch point. Pick an open neighbourhood V of y such that $f^{-1}(V)$ is a disjoint union of open subsets U such that $f|_U : U \longrightarrow V$ is a homeomorphism. Possibly shrinking V, we may assume that U is part of an atlas and this defines an atlas on V.

Since the deck transformations are holomorphic, and the branch point belong to only one chart, this atlas on Y has holomorphic transition functions. This makes Y into a Riemann surface and f into a holomorphic map. □

Definition 18.8. *Let Y be a connected topological space. A continuous map $f : X \longrightarrow Y$ is called the* ***universal cover*** *of Y if X is simply connected and f is an unramified cover.*

As the name might suggest the universal cover comes with a universal property. Suppose that $f : X \longrightarrow Y$ is the universal cover. If $g : Z \longrightarrow Y$ is an unramified cover then there is a unique continuous map $h : X \longrightarrow Y$ such that $f = g \circ h$. In particular h is an unramified cover. By virtue of the universal property, the universal cover is unique up to unique isomorphism.

Suppose that we pick a point $b \in Y$. Pick a point $a \in X$ over b. Then we may lift any path $\gamma : [0, 1] \longrightarrow Y$ starting at b to a path $\psi : [0, 1] \longrightarrow X$. If $y = \gamma(1)$ then note that $x = \psi(1)$ belongs to the fibre over $y \in Y$. Suppose that $\gamma' \sim \gamma$ is homotopic to γ, where $\gamma'(0) = b$ and $\gamma'(1) = y$. Let ψ' be the lift of γ'. Since we can lift homotopies, it follows that $\psi' \sim \psi$ and so $\psi'(1) = x$. Thus the point x only depends on the homotopy class of γ.

Conversely given any point x of the fibre over y, then we may find a path ψ connecting a to x. The composition of ψ with f gives a path γ connecting a to y. If ψ' is another path in X with endpoint x, then we may find a homotopy in X between ψ and ψ', as X is simply connected. If γ' is the composition of ψ' and f, it follows that γ and γ' are homotopic. Thus the points of X are in bijection with the homotopy classes of paths starting at b (so that this gives a way to construct X).

Now suppose that pick an element $\phi \in G = \pi_1(Y, b)$. Then $\gamma' = \gamma \cdot \phi$ is another path in Y connecting b to y. Lifting γ and γ' to paths ψ and ψ' with endpoints x and x' this defines an action

$$X \times G \longrightarrow X \qquad \text{by the rule} \qquad x \longrightarrow x \cdot g$$

In fact this action is faithful and actually realises X as a G-bundle over Y (i.e locally over Y, $X \simeq Y \times G$ and this homeomorphism respects the action). Put differently $Y = X/G$. The intermediate covers $g\colon Z \longrightarrow Y$ are then given by subgroups H of G, where $Z = X/H$.

Putting all of the material together in the previous two sections, we get:

Theorem 18.9. *Let Y be a Riemann surface.*

Then $Y \simeq X/G$, where X is a simply connected Riemann surface and G is a group of automorphisms of X which acts on X properly discontinuously, such that the quotient map $f\colon X \longrightarrow Y$ is an unramified cover.

Proof. Let $f\colon X \longrightarrow Y$ be the universal cover of Y. Then f is continuous, $Y = X/G$, where $G \simeq \pi_1(X, x)$ acts on X properly discontinuously. Note that G is the group of deck transformations of f.

We may make X into a Riemann surface in such a way that f is holomorphic. But then every element of X is then biholomorphic, so that Y is the quotient Riemann surface. □

Example 18.10. *Let $\Lambda \subset \mathbb{C}$ be a lattice. That is $\Lambda \simeq \mathbb{Z}^2$ as groups and the span of Λ over $\mathbb{R}$ is the whole of $\mathbb{C}$. Then the quotient*

$$E = \frac{\mathbb{C}}{\Lambda},$$

*is a Riemann surface, which is known as an **elliptic curve**. In fact the quotient is homeomorphic to $S^1 \times S^1$, so that E is an example of a compact Riemann surface.*

In fact the topological classification of connected, compact oriented surfaces is easy. Given any oriented surface S, one can construct another by adding a handle. The compact oriented surfaces are then classified by the number of handles g one adds to S^2. In terms of the universal cover there are then three possibilities. If $g = 0$, then $S \simeq S^2$. As S^2 is simply connected, it is its own universal cover. If $g = 1$ then $S \simeq S^1 \times S^1$. Since the universal cover of S^1 is $\mathbb{R}$ and since the universal cover of a product is the product of the universal covers (by the universal property of the universal cover and the product) it follows that $S^1 \times S^1$ has universal cover $\mathbb{R}^2 \simeq \mathbb{C}$. Finally the same holds for higher genus (although perhaps it is better to say that the universal cover is the disc).

It is therefore interesting to classify all simply connected Riemann surfaces:

Theorem 18.11. *Let X be a simply connected Riemann surface. Then X is biholomorphic to one of*

(1) $\mathbb{P}^1$

(2) $\mathbb{C}$*, and*
(3) the unit disc Δ*.*

Note that these three Riemann surfaces are not isomorphic to each other. Note also that we have a complete classification of the automorphism group in each case:

(1) $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$, the group of Möbius transformations
$$z \longrightarrow \frac{az+b}{cz+d}.$$

(2) $\mathrm{Aut}(\mathbb{C})$ is the subgroup of the Möbius transformations which fix ∞,
$$z \longrightarrow az+b.$$

(3) Finally $\mathrm{Aut}(\Delta)$ is also given by a subgroup of the Möbius transformations. Every automorphism has the form
$$z \longrightarrow e^{i\theta}\frac{z-a}{1-\bar{a}z},$$
where $a \in \Delta$ and $\theta \in [0, 2\pi)$.