4. Poisson formula

In fact we can write down a formula for the values of u in the interior using only the values on the boundary, in the case when E is a closed disk. First note that (3.5) determines the value at the origin. On the other hand, we can apply a Möbius transformation to move any point to the centre. Suppose that u is harmonic on the circle $|z| \leq R$. Then the fractional linear transformation

$$z = S(w) = \frac{R(Rw + a)}{R + \bar{a}w},$$

carries the circle $|w| \leq 1$ onto the circle $|z| \leq R$ and sends $w = 0$ to $z = a$.

The function $u(S(w))$ is harmonic in the unit circle $|w| \leq 1$ and we obtain

$$u(a) = \frac{1}{2\pi} \int_{|w|=1} u(S(w))\, d \arg w.$$

As

$$w = \frac{R(z - a)}{R^2 - \bar{a}z},$$

we see that

$$d \arg w = -i\frac{dw}{w} = -i\left(\frac{1}{z - a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right) \mathrm{d}z = \left(\frac{z}{z - a} + \frac{\bar{a}z}{R^2 - \bar{a}z}\right) \mathrm{d}\theta.$$

Now $R^2 = z\bar{z}$ on the circle $|z| = R$, so that the last expression in brackets can be rewritten as

$$\frac{z}{z - a} + \frac{\bar{a}}{\bar{z} - \bar{a}} = \frac{R^2 - |a|^2}{|z - a|^2}.$$

Equivalently we have

$$\frac{1}{2}\left(\frac{z + a}{z - a} + \frac{\bar{z} + \bar{a}}{\bar{z} - \bar{a}}\right) = \operatorname{Re} \frac{z + a}{z - a}.$$

Thus

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(z)\, d\theta = \frac{1}{2\pi} \int_{|z|=R} \operatorname{Re} \frac{z + a}{z - a} u(z)\, d\theta.$$

These expressions are known as Poisson's formula. In polar coordinates we have

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \phi) + r^2} u(Re^{i\theta})\, d\theta.$$

The derivation of this formula used the fact that $u(z)$ is harmonic on the whole closed disk. Suppose that we weaken the hypothesis so that $u(z)$ is only harmonic on the interior and continuous on the closed disk.

Then $u(rz)$ is harmonic on the closed disk, for any $0 < r < 1$ and by what we have already proved, we obtain

$$u(ra) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(rz)\, d\theta.$$

Now let r tend to one. Note that $u(rz)$ tends uniformly to $u(z)$, since the disk $|z| \leq R$ is compact. We have thus proved:

Theorem 4.1 (Poisson's Formula)**.** *Suppose that $u(z)$ is harmonic for $|z| < R$ and continuous for $|z| \leq R$. Then*

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\, d\theta,$$

for all $|a| < R$.

We have

$$u(z) = \operatorname{Re}\left[\frac{1}{2\pi i} \int_{|w|=R} \frac{w+z}{w-z} u(w) \frac{dw}{w}\right].$$

Since the expression in brackets is holomorphic for $|z| < R$, we have that $u(z)$ is the real part of the holomorphic function

$$f(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{w+z}{w-z} u(w) \frac{dw}{w} + iC,$$

where C is an arbitrary real constant. This is known as Schwarz's formula.

Note that if we apply (4.1) to the harmonic function $u = 1$ then we get

$$\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2}\, d\theta = 2\pi,$$

for all $|a| < R$.

Now suppose that u is no longer defined on the interior, it is only defined on the boundary and it not even continuous on the boundary, suppose for example that it is only a piecewise continuous function on the boundary.

Then the integral in (4.1) still makes sense and in fact the integral is still the real part of an analytic function, so that the integral is still a harmonic function. However we no longer know what happens on the boundary.

Suppose that we set $R = 1$. Let $U(\theta)$ be a piecewise continuous function on the interval $0 \leq \theta \leq 2\pi$.

Definition 4.2. *The integral*

$$P_U(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z} U(\theta)\, d\theta,$$

is called the ***Poisson integral****.*

Note that the Poisson integral is a function of U, which is linear in U,

$$P_{U+V} = P_U + P_V \qquad \text{and} \qquad P_{cU} = cP_U.$$

We have $U \geq 0$ implies that $P_U \geq 0$. Thus P is a positive linear functional. Note that $P_c = c$. In particular

Lemma 4.3.

$$m \leq U \leq M \qquad \textit{implies that} \qquad m \leq P_U \leq M.$$

Proof. By assumption $U - m \geq 0$. Thus

$$P_U - m = P_U - P_m = P_{U-m} \geq 0$$

and so $m \leq P_U$. The other inequality is similar. $\square$

Theorem 4.4. *$P_U(z)$ is harmonic for $|z| < 1$ and*

$$\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0),$$

if U is continuous at θ_0.

Proof. We have already seen that P_U is harmonic. Pick complementary arcs C_1 and C_2 and denote by U_i the function which is zero on C_{3-i} and is equal to U on C_i. Then $U = U_1 + U_2$ so that

$$P_U = P_{U_1} + P_{U_2}.$$

Note that P_{U_i} is given by a line integral over the arc C_i. Thus P_{U_i} is continuous except possibly along C_i. Now

$$\operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z} = \frac{1 - |z|^2}{|e^{i\theta} - z|^2},$$

vanishes on $|z| = 1$ for $z \neq e^{i\theta}$. Thus P_{U_1} is zero on the interior of the arc C_2 and by continuity $P_{U_1}(z) \to 0$ as $z \to e^{i\theta} \in C_2$.

Replacing U by $U - U(\theta_0)$ we may suppose that $U(\theta_0) = 0$. Given $\epsilon > 0$ we can find C_1 and C_2 such that $e^{i\theta_0}$ is an interior point of C_2 and $|U(\theta)| < \epsilon/2$ for $e^{i\theta} \in C_2$. But then $|U_2(\theta)| < \epsilon/2$ for all θ so that $|P_{U_2}(z)| < \epsilon/2$ for all $|z| < 1$. On the other hand, since U_1 is piecewise continuous and $U_1(\theta_0) = 0$, there is a constant $\delta > 0$ such that

$$|P_{U_1}(z)| < \epsilon/2 \qquad \text{for} \qquad |z - e^{i\theta_0}| < \delta.$$

But then

$$|P_U(z)| \leq |P_{U_1}(z)| + |P_{U_2}(z)| < \epsilon,$$

as soon as $|z| < 1$ and $|z - e^{i\theta_0}| < \delta$. □

Theorem 4.5 (Reflection Principle)**.** *Let U be a region which is invariant under complex conjugation, let U^+ be the part above the real axis and let σ be the intersection of U with the real axis.*

Let $v(z)$ be a continuous function on $U^+ \cup \sigma$ which is harmonic on U^+ and zero on σ. Then $v(z)$ can be extended to a harmonic function on the whole of U and this function satisfies

$$v(\bar{z}) = -v(z).$$

Further if $v(z)$ is the imaginary part of a holomorphic function $f(z)$ in U^+ then $f(z)$ can be extended to a holomorphic function on the whole of U and this function satisfies $f(z) = \bar{f}(\bar{z})$.

Proof. Define a function $V \colon U \longrightarrow \mathbb{R}$ by the rule

$$V(z) = \begin{cases} v(z) & \text{if } \operatorname{Im} z > 0 \\ 0 & \text{if } \operatorname{Im} z = 0 \\ -v(\bar{z}) & \text{if } \operatorname{Im} z < 0. \end{cases}$$

It suffices to show that $V(z)$ is harmonic in a neighbourhood of a point $x_0 \in \sigma$. Pick a small disc about x_0 contained in U and let P_V be the Poisson integral with respect to this disk, using the boundary values given by V.

Now the difference $V - P_V$ is harmonic on the upper half disc B^+. It vanishes on the upper circle by (4.4) and also on the diameter, since V is zero on σ and P_V is zero by symmetry. Thus by the maximum and minimum principle $V = P_V$ on B^+. Similarly $V = P_V$ on the lower half disc B^-. Thus $V = P_V$ is harmonic on the whole of B. Thus $V(z)$ is harmonic on the whole of U.

We may construct a harmonic conjugate $-u_0$ of v in the disc B. We may normalise u_0 so that $u_0 = \operatorname{Re} f(z)$ on B^+. Consider

$$U_0(z) = u_0(z) - u_0(\bar{z}).$$

On σ,

$$\frac{\partial U_0}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial U_0}{\partial y} = 2\frac{\partial u_0}{\partial y} = -2\frac{\partial v}{\partial x} = 0.$$

Thus the analytic function

$$\frac{\partial U_0}{\partial x} - i\frac{\partial U_0}{\partial y}$$

vanishes on the real axis so that it vanishes everywhere. Thus U_0 is a constant which must be zero. In particular $u_0(z) = u_0(\bar{z})$. Repeating this construction for any point of σ and observing that the functions so constructed agree on overlaps, the result follows. $\square$