7. Principal values

Consider an integral of the form

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x,$$

where $R(x)$ is a rational function. Assume that the degree of the denominator is at least two more than the degree of the numerator. If we split this integral into real and imaginary parts we get two integrals:

$$\int_{-\infty}^{\infty} R(x)\cos x\,\mathrm{d}x, \qquad \text{and} \qquad \int_{-\infty}^{\infty} R(x)\sin x\,\mathrm{d}x.$$

To compute the original integral, as usual we integrate over a closed path which is the interval from $-R$ to R and a semicircle of radius R in the upper half plane. As before the integral over the semicircle goes to zero as R goes to infinity. It then follows by the residue theorem that

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz}.$$

Now suppose that $R(z)$ has only a simple zero at ∞, that is, suppose the denominator has degree exactly one more than the numerator. In this case we have to be a little bit more careful in our choice of contour. It is not so convenient to use semicircles. For a start the integral over the semicircle is not so easy to estimate; secondly it is also not so clear that the real integral converges, so not only do we need to show convergence of the limit

$$\lim_{X\to\infty} \int_{-X}^{X} R(x)e^{ix}\,\mathrm{d}x,$$

but also of the integral

$$\int_{-X_1}^{X_2} R(x)e^{ix}\,\mathrm{d}x,$$

as X_1 and X_2 approach infinity independently.

The solution is to integrate around a rectangle with vertices X_2, $X_2 + iY$, $-X_1 + iY$ and $-X_1$, where $Y > 0$. If X_1, X_2 and Y are sufficiently large then this rectangle contains all of the poles in the upper half plane. By hypothesis $|zR(z)|$ is bounded, so that the integral over the right vertical side is, except for a constant factor, less than,

$$\int_0^Y e^{-y}\frac{\mathrm{d}y}{|z|} < \frac{1}{X_2}\int_0^Y e^{-y}\,\mathrm{d}y.$$

The last integral can be computed and it is less than one. Thus the integral over the right vertical side is less than a constant multiple of $1/X_2$. Similarly for the integral over the left vertical side.

The integral over the top horizontal side is less than

$$\frac{e^{-Y}(X_1 + X_2)}{Y}$$

multiplied by a constant. If we fix X_1 and X_2 and let Y go to infinity this goes to zero and so

$$\left| \int_{-X_1}^{X_2} R(x)e^{ix}\,\mathrm{d}x - 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz} \right| < A(\frac{1}{X_1} + \frac{1}{X_2})$$

for some constant A. Thus

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz}.$$

So far we have been tacitly assuming that there are no poles along the real axis. But suppose that there are. Note that the real or imaginary part of the integral might well still exist if the poles are at the zeroes of $\cos x$ or of $\sin x$.

For example, let's suppose that $R(z)$ has a simple pole at the origin and nowhere else on the real axis. Take a contour which is a rectangle and a small circle of radius ρ which goes below the x-axis. If X_1, X_2 and Y are sufficiently large and ρ is sufficiently small then this contour includes all of the poles in the upper half plane, the pole at the origin and nothing else.

Suppose that the residue at the origin is B so that

$$R(z)e^{iz} = B/z + R_0(z),$$

where $R_0(z)$ is holomorphic at the origin. The integral of the first term is $\pi i B$ and the integral of the second term tends to zero as ρ tends to zero.

Thus

$$\lim_{\rho \to 0} \left(\int_{-\infty}^{-\rho} + \int_{\rho}^{\infty} R(x)e^{ix}\,\mathrm{d}x \right) = 2\pi i \left[\sum_{y>0} \operatorname{Res} R(z)e^{iz} + B/2 \right].$$

The limit on the left is called **Cauchy's principal value**. It exists even though the integral itself might have no meaning. It is as though one half of the residue at zero has been included in the contour integral.

In the general case where there are several simple poles along the real axis, we have

$$\text{pr. v.} \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz} + \pi i \sum_{y=0} \operatorname{Res} R(z)e^{iz}.$$

For example,

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,\mathrm{d}x = \pi i.$$

If we separate into real and imaginary parts, we see that the real part is trivial since the integrand is odd. For the imaginary part, we don't need to take the principal value and we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$