MIDTERM
MATH 220B, UCSD, WINTER 15

You have 80 minutes.

There are 5 problems, and the total number of points is 80. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:________________________________

Signature:______________________________

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	10	
7	10	
Total	80	

1. (15pts) (i) Let $R(z)$ be a rational function with a simple pole at zero and no other poles along the real axis. How does one define the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x?$$

Solution:

$$\text{pr. v.} \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = \lim_{\rho\to 0}\int_{-\infty}^{\rho} R(x)e^{ix}\,\mathrm{d}x + \lim_{\rho\to 0}\int_{\rho}^{\infty} R(x)e^{ix}\,\mathrm{d}x.$$

(ii) Suppose that $R(z)$ is zero at infinity and has simple poles along the real axis. State a version of the residue theorem for

$$\text{pr. v.} \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x.$$

Solution:

$$\text{pr. v.} \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = 2\pi i\left[\sum_{y>0} \operatorname{Res} R(z)e^{iz} + \frac{1}{2}\sum_{y=0} \operatorname{Res} R(z)e^{iz}\right].$$

(iii) Calculate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x.$$

Solution: $\sin z/z$ has a single simple pole at 0 with residue 1. Since the integral converges:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \operatorname{Im} \text{pr. v.} \int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x = \pi.$$

2. (15pts) (i) show that $\log r$ is a harmonic function on $U = \mathbb{C}^*$.

Solution: $\log r$ is certainly $\mathcal{C}^\infty$.

$$r\frac{\partial}{\partial r}\left(r\frac{\partial \log r}{\partial r}\right) + \frac{\partial^2 \log r}{\partial \theta^2} = 0.$$

Thus $\log r$ is harmonic.

(ii) Show that if u is a harmonic function on a region U then

$$f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

is a holomorphic function on U.

Solution: By assumption $U = \frac{\partial u}{\partial x}$ and $V = \frac{\partial u}{\partial y}$ are $\mathcal{C}^1$. We have

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial V}{\partial y} \\ \frac{\partial U}{\partial y} &= \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial V}{\partial x}.\end{aligned}$$

so that U and V satisfy the Cauchy-Riemann equations. Thus f is holomorphic.

(iii) Give an example of a harmonic function u and a region U such that u is not the real part of a holomorphic function f on U.

Solution: Let $u = \log r$ and $U = \mathbb{C}^*$. If $f(z)$ is any holomorphic function whose real part is $\log r$ then $f(z)$ is a holomorphic branch of the logarithm. On the other hand there is no holomorphic branch of the logarithm on the whole of U, since the exponential function is not injective.

3. (15pts) (i) Define the conjugate differential $*\mathrm{d}u$ of a harmonic function u on a region U.

Solution:

$$*\mathrm{d}u = -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y.$$

(ii) Show that if u_1 and u_2 are harmonic functions on a region U and γ is a cycle homologous to zero then

$$\int_\gamma u_1 *\mathrm{d}u_2 - u_2 *\mathrm{d}u_1 = 0.$$

Solution:

It suffices to prove this when $\gamma = \partial R$ is the boundary of a rectangle R. In this case we may find conjugate harmonic functions v_1 and v_2. In this case

$$u_1 *\mathrm{d}u_2 - u_2 *\mathrm{d}u_1 = u_1\,\mathrm{d}v_2 - u_2\,\mathrm{d}v_1 = u_1\,\mathrm{d}v_2 + v_1\,\mathrm{d}u_2 - \mathrm{d}(u_2 v_1).$$

Now $\mathrm{d}(u_2v_1)$ is an exact differential and $u_1\,\mathrm{d}v_2+v_1\,\mathrm{d}u_2$ is the imaginary part of

$$(u_1 + iv_1)\,\mathrm{d}(u_2 + iv_2).$$

Integrating an exact differential over γ is zero. On the other hand

$$(u_1 + iv_1)\,\mathrm{d}(u_2 + iv_2) = F(z)f(z)\,\mathrm{d}z$$

for appropriate holomorphic functions $F(z)$ and $f(z)$, so that integrating the product above is zero, since the product of two holomorphic functions is holomorphic.

(iii) Show that if u_1 and u_2 are harmonic functions on an annulus $\rho_1 < |z| < \rho_2$ then there are constants α and β such that

$$\int_{|z|=r} u \, \mathrm{d}\theta = \alpha \log r + \beta.$$

Solution:
Let γ be the difference of two circles of radius r_1 and r_2 between ρ_1 and ρ_2. Then γ is homologous to zero. Take $u_1 = u$ and $u_2 = \log r$. On a circle of radius r_i we have

$$u_1 *\mathrm{d}u_2 - u_2 *\mathrm{d}u_1 = (u - r_i \log r_i \frac{\partial u}{\partial r})\mathrm{d}\theta.$$

Thus

$$\int_{|z|=r_i} (u - r_i \log r_i \frac{\partial u}{\partial r})\mathrm{d}\theta$$

is independent of r_i. Similarly

$$\int_{|z|=r_i} r_i \frac{\partial u}{\partial r} \, \mathrm{d}\theta = \int_{|z|=r_i} *\mathrm{d}u$$

is independent of r_i.

(iv) If u is a harmonic function on a closed disc $|z - z_0| = r$ of radius r centred at z_0 then show that

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta,$$

for any $z_0 \in U$, where r is sufficiently small.

Solution: By (iii)

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta = \alpha \log r + \beta.$$

for constants α and β. As u is defined on some small disc centred at z_0, we must have $\alpha = 0$. $u(z_0) = \beta$ by continuity as we let r go to zero.

4. (15pts) (i) Show that the Möbius transformation

$$z = S(w) = \frac{R(Rw+a)}{R+\bar{a}w},$$

carries the circle $|w| \leq 1$ onto the circle $|z| \leq R$ and sends $w = 0$ to $z = a$.

Solution: Surely $S(0) = a$. S corresponds to the matrix

$$\frac{1}{R}\begin{pmatrix} R^2 & Ra \\ \bar{a} & R \end{pmatrix} = \begin{pmatrix} R & a \\ \bar{a}/R & 1 \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a/R \\ \bar{a}/R & 1 \end{pmatrix}.$$

The first matrix of the last product corresponds to the Möbius transformation $z \longrightarrow Rz$. Replacing R by 1 and a by a/R we may therefore assume that $R = 1$.

Suppose that $|w| = 1$. Then

$$\begin{aligned} \left|\frac{w+a}{1+\bar{a}w}\right| &= |w| \cdot \left|\frac{w+a}{w^{-1}+\bar{a}}\right| \\ &= \left|\frac{w+a}{\bar{w}+\bar{a}}\right| \\ &= 1. \end{aligned}$$

Thus S maps the circle $|w| = 1$ to the circle $|z| = 1$.

(ii) Show that if u is a harmonic function on a circle $|z| \leq R$ then

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\, d\theta.$$

Solution: Let $v(w) = u(S(w))$. Then v is a harmonic function. Start with the 3 (iv) applied to $v(w)$, $w_0 = 0$ and $r = 1$,

$$u(a) = v(0) = \frac{1}{2\pi} \int_{|w|=1} v(w)\, \mathrm{d} \arg w.$$

Note that

$$w = \frac{R(z-a)}{R^2 - \bar{a}z}.$$

Then

$$\mathrm{d} \arg w = \frac{1}{i} \frac{\mathrm{d}w}{w} = \frac{1}{i} \left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z} \right) \mathrm{d}z = \left(\frac{z}{z-a} + \frac{\bar{a}z}{R^2 - \bar{a}z} \right) \mathrm{d}\theta.$$

Note that as $R^2 = z\bar{z}$ on the circle $|z| = R$ the expression in brackets is

$$\frac{z}{z-a} + \frac{\bar{a}}{\bar{z} - \bar{a}} = \frac{R^2 - |a|^2}{|z-a|^2}.$$

Putting all of this together gives the result.

5. (20pts) (i) Show that the second logarithmic derivative of the Gamma function

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

is

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Solution: We first calculate the logarithmic derivative

$$\begin{aligned}\frac{\Gamma'(z)}{\Gamma(z)} &= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1/n}{1+\frac{z}{n}} \\ &= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+z}.\end{aligned}$$

Since the product is absolutely convergent we can differentiate term by term. If we differentiate again we get

$$\begin{aligned}\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) &= \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(n+z)^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}.\end{aligned}$$

(ii) Show that

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2).$$

Solution: We have

$$\begin{aligned}
\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right) &= \sum_{n=0}^{\infty}\frac{1}{(z+n)^2} + \sum_{n=0}^{\infty}\frac{1}{(z+n+1/2)^2} \\
&= 4\left[\sum_{n=0}^{\infty}\frac{1}{(2z+2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2z+2n+1)^2}\right] \\
&= 4\sum_{m=0}^{\infty}\frac{1}{(2z+m)^2} \\
&= 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right).
\end{aligned}$$

If we integrate then we get

$$\Gamma(z)\Gamma(z+1/2) = e^{az+b}\Gamma(2z),$$

where a and b are constants to be determined. Substituting $z = 1/2$ and $z = 1$ we make use of the known values

$$\Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(1) = 1, \qquad \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}, \qquad \text{and} \qquad \Gamma(2) = 1.$$

This gives

$$\begin{aligned}
a/2 + b &= 1/2\log\pi \\
a + b &= 1/2\log\pi - \log 2.
\end{aligned}$$

It follows that

$$a = -2\log 2 \qquad \text{and} \qquad b = 1/2\log\pi + \log 2.$$

Bonus Challenge Problems

6. (10pts) State and prove Jensen's theorem.

7. (10pts) State and prove the formula of Gauss for the Gamma function.