MODEL ANSWERS TO THE FIRST HOMEWORK

1. Note that

$$1 - \frac{1}{n^2} = \frac{n^2-1}{n^2} = \frac{(n-1)(n+1)}{n \cdot n}.$$

It follows that the partial product is

$$\begin{aligned} p_n &= (1-1/4)(1-1/9)\dots(1-1/(n-1)^2)(1-1/n^2) \\ &= \frac{1\cdot 3}{2\cdot 2}\frac{2\cdot 4}{3\cdot 3}\frac{3\cdot 5}{4\cdot 4}\dots\frac{(n-2)n}{(n-1)\cdot(n-1)}\frac{(n-1)(n+1)}{n\cdot n} \\ &= \frac{1}{2}\frac{n+1}{n}. \end{aligned}$$

Therefore

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \lim_{n\to\infty} p_n = \frac{1}{2}.$$

2. The zeroes of $\cos\sqrt{z}$ are $a_n = (2n+1)^2\pi^2/4$. Consider the product

$$\prod_{n=0}^{\infty}\left(1-\frac{z}{a_n}\right).$$

The product converges absolutely if and only if

$$\sum_{n=0}^{\infty}\frac{|z|}{|a_n|}$$

converges. As

$$\sum \frac{1}{(2n+1)^2},$$

converges it follows that

$$\prod_{n=0}^{\infty}\left(1-\frac{z}{a_n}\right)$$

is the canonical product for $\cos\sqrt{z}$. The genus of the canonical product is zero.
We have that

$$\cos\sqrt{z} = e^{g(z)}\prod_{n=0}^{\infty}\left(1-\frac{z}{a_n}\right)$$

for some entire function $g(z)$. The genus of $\cos\sqrt{z}$ is then the degree of $g(z)$, if $g(z)$ is a polynomial and ∞ otherwise. If we replace $\sqrt{z}$ by z we get

$$\cos z = e^{g(z^2)} \prod_{n=0}^{\infty} \left(1 - \frac{z^2}{a_n}\right).$$

Let's replace z by $\pi/2 - z$ and check we get the canonical product for $\sin z$:

$$\begin{aligned}
\sin z &= \cos(\pi/2 - z) \\
&= e^{g((\pi/2-z)^2)} \prod_{n=0}^{\infty} \left(1 - \frac{(\pi/2 - z)}{(2n+1)\pi/2}\right)\left(1 + \frac{(\pi/2 - z)}{(2n+1)\pi/2}\right) \\
&= e^{g((\pi/2-z)^2)} \prod_{n=0}^{\infty} \left(\frac{n\pi + z}{(2n+1)\pi/2}\right)\left(\frac{(n+1)\pi - z)}{(2n+1)\pi/2}\right) \\
&= ze^{g((\pi/2-z)^2)} \prod_{n\neq 0}^{\infty} \frac{4n(n+1)}{(2n+1)^2}\left(1 + \frac{z}{n\pi}\right)\left(1 - \frac{z}{n\pi}\right) \\
&= cze^{g((\pi/2-z)^2)} \prod_{n\neq 0}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right),
\end{aligned}$$

for some constant c. Comparing with the canonical product for $\sin z$ we see that $g((\pi/2 - z)^2)$ is constant. Thus $g(z)$ has degree zero and the genus of $\cos\sqrt{z}$ is zero.

3. Note that the complement in $\mathbb{P}^1$ of the unbounded region contains at least one bounded region and the point at ∞, which is an isolated point of the complement. Thus the complement of the unbounded region is not connected and so the unbounded region is not simply connected. On the other hand the boundary of every bounded region is a simply closed curve. The Jordan curve theorem implies that the complement of the bounded region in $\mathbb{C}$ is connected Since the point ∞ is in the closure of the complement, the complement in $\mathbb{P}^1$ also has only one component. Thus every bounded region is simply connected.

4. Fix a point $a \in U$, and for any point $z \in U$, pick a path γ from a to z. Then define

$$g(z) = \int_\gamma \frac{\mathrm{d}z}{z}.$$

Note that as U is simply connected, if γ_1 and γ_2 are two closed paths starting at a and ending at z then for the cycle $\gamma_1 - \gamma_2$ we have

$$\int_{\gamma_1-\gamma_2} \frac{dz}{z} = 0.$$

But this exactly says,

$$\int_{\gamma_1} \frac{dz}{z} = \int_{\gamma_2} \frac{dz}{z}.$$

Thus g is well-defined. It is clear that $g(z)$ is holomorphic and

$$g'(z) = \frac{1}{z}.$$

We have

$$\begin{aligned}\frac{d}{dz}(ze^{-g(z)}) &= e^{-g(z)} - e^{-g(z)} \\ &= 0.\end{aligned}$$

Thus $ze^{-g(z)}$ is constant. It follows that

$$e^{g(z)+\log z_0 - g(z_0)} = z,$$

so that the logarithm exists and is holomorphic. But then we may define

$$z^\alpha = e^{\alpha \log z} \qquad \text{and} \qquad z^z = e^{z \log z}.$$

5. We calculate the second logarithmic derivative of $\Gamma(z)$

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{m=0}^{\infty} \frac{1}{(z+m)^2}.$$

Let's compare this with the second logarithmic derivative of

$$\Phi(z) = \Gamma\left(\frac{z}{n}\right)\Gamma\left(\frac{z+1}{n}\right)\cdots\Gamma\left(\frac{z+n-1}{n}\right).$$

$$\begin{aligned}\frac{d}{dz}\left(\frac{\Phi'(z)}{\Phi(z)}\right) &= \frac{1}{n^2}\sum_{m=0}^{\infty} \frac{1}{(z/n+m)^2} + \frac{1}{((z+1)/n+m)^2} + \cdots + \frac{1}{((z+n-1)/n+m)^2} \\ &= \sum_{m=0}^{\infty} \frac{1}{(z+mn)^2} + \frac{1}{(z+mn+1)^2} + \cdots + \frac{1}{(z+mn+n-1)^2} \\ &= \frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right).\end{aligned}$$

If we integrate both sides we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = \frac{\Phi'(z)}{\Phi(z)} + a$$

Integrating once more we get

$$\log \Gamma(z) = \log \Phi(z) + az + b.$$

Thus

$$\Gamma(z) = e^{az+b}\Phi(z),$$

where a and b are constants to be determined.
Note that

$$\Phi(2) = \frac{\Phi(1)\Gamma(1+1/n)}{\Gamma(1/n)} = \frac{\Phi(1)}{n}.$$

Thus

$$1 = \frac{\Gamma(2)}{\Gamma(1)} = e^a\frac{\Phi(2)}{\Phi(1)} = \frac{e^a}{n}.$$

It follows that

$$e^{az} = n^z.$$

Assume that $n = 2m+1$ is odd (the case where n is even can be treated similarly or one can apply induction and Legendre's duplication formula). To determine the constant b we need to determine the value of one value of Φ.

$$\begin{aligned}\Phi(1) &= \prod_{i=1}^{n-1}\Gamma(i/n)\Gamma(1)\\ &= \prod_{i=1}^{m}\Gamma(i/n)\Gamma((n-i)/n)\\ &= \prod_{i=1}^{m}\frac{\pi}{\sin \pi i/n},\end{aligned}$$

where we used the functional equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Let $\omega = e^{2\pi i/n}$, a primitive root of unity. Then

$$\sin \pi i/n = \frac{1}{2}\,\mathrm{Im}(\omega^i - \omega^{-i}),$$

as $\bar{\omega} = \omega^{-1}$. Thus

$$\prod_{i=1}^{m}\frac{\pi}{\sin \pi i/n} = \frac{2^m\pi^m}{\prod_{i=1}^{m}\mathrm{Im}(\omega^i - \omega^{n-i})}.$$

To finish we just need to show that

$$\prod_{i=1}^{m}\mathrm{Im}(\omega^i - \omega^{n-i}) = \sqrt{n}.$$

As it is not hard to see that the product is positive, it suffices to show that

$$\prod_{i=1}^{n-1}(\omega^i - \omega^{n-i}) = n.$$

Now

$$\begin{aligned}\prod_{i=1}^{n-1}(\omega^i - \omega^{n-i}) &= \prod_{i=1}^{n-1}\omega^i \prod_{i=1}^{n-1}(1-\omega^i) \\ &= \omega^{\sum_{i=1}^{n-1} i}\prod_{i=1}^{n-1}(1-\omega^i) \\ &= \prod_{i=1}^{n-1}(1-\omega^i).\end{aligned}$$

Now

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x-1} = \prod_{i=1}^{n-1}(x - \omega^i).$$

Thus, setting $x = 1$, we get

$$\prod_{i=1}^{m}(1-\omega^i) = 1 + 1 + \cdots + 1 = n.$$

Putting all of this together we get

$$(2\pi)^{\frac{n-1}{2}}\Gamma(z) = n^{(z-1/2)}\Gamma\left(\frac{z}{n}\right)\Gamma\left(\frac{z+1}{n}\right)\cdots\Gamma\left(\frac{z+n-1}{n}\right).$$

6. As

$$\Gamma(z+1) = z\Gamma(z)$$

by induction on n we have

$$\Gamma(z+n+1) = (z+n)(z+n-1)\cdots z\Gamma(z).$$

Thus

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{(z+n)(z+n-1)\cdots z},$$

Therefore $\Gamma(z)$ has a pole order one $z = 0$, $z = -1$, $\ldots$, $z = -n$. The value of the pole at $z = -n$ is then

$$\lim_{z\to -n}(z+n)\Gamma(z) = \lim_{z\to -n}\frac{\Gamma(z+n+1)}{(z+n-1)\cdots z} = \frac{\Gamma(1)}{-1\cdot -2\cdots - n} = \frac{(-1)^n}{n!}.$$