MODEL ANSWERS TO THE THIRD HOMEWORK

1. Since f is real on the real axis, we have

$$f(\bar{z}) = \bar{f}(z),$$

by the reflection principle. On the other hand, as f is purely imaginary on the imaginary axis, $g(z) = if(-iz)$ is real on the real axis, so that g also satisfies

$$g(\bar{z}) = \bar{g}(z),$$

by the reflection principle. It follows that

$$f(-x + iy) = -u + iv$$

where $f(z) = u + iv$. Thus

$$f(-z) = f(-x - iy) = \bar{f}(-x + iy) = -f(x + iy) = -f(z),$$

so that f is odd.

2. (i) Let

$$R(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}.$$

Then $R(z)$ is a rational function with a double zero at infinity. The zeroes of the denominator are

$$0 = z^4 + 10z^2 + 9 = (z^2 + 1)(z^2 + 9),$$

so that $R(z)$ has simple poles at $z = \pm i$ and $z = \pm 3i$. We integrate $R(z)$ around the standard contour from $-R$ to R along the real axis and along a semicircle of radius R in the upper half plane. As the radius R tends to infinity the integral along the semicircle goes to zero. If $R > 3$ then the contour goes around all poles in the upper half plane which are at i and $3i$. The residue at i is

$$\frac{i^2 - i + 2}{(i + i)(i^2 + 9)} = \frac{1 - i}{16i}.$$

The residue at $3i$ is

$$\frac{(3i)^2 - 3i + 2}{(3i + 3i)((3i)^2 + 1)} = \frac{7 + 3i}{48i}.$$

Taking the limit as R goes to infinity by the residue theorem we get

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,\mathrm{d}x = 2\pi i \left(\frac{1 - i}{16i} + \frac{7 + 3i}{48i} \right) = \frac{5\pi}{12}.$$

(ii) This is done in the model answers to the final of 220A but just for practice let's pick a different contour. We start just above the real axis go around a big circle of radius R centred at the origin end just below the real axis go parallel to the real axis almost to the origin, describe a small circle of radius ρ centred at the origin clockwise and go back along the x-axis.
It is straightforward to check that the integral along the big circle goes to zero as R goes to infinity and the integral along the small circle also goes to zero as ρ goes to zero.
Let

$$I = \int_0^\infty \frac{x^{1/3}\,\mathrm{d}x}{1+x^2}.$$

We define $z^{1/3}$ using a branch of the logarithm which excludes the positive real axis and the argument θ satisfies $0 < \theta < 2\pi$. If $z = re^{i\theta}$ then $z^{1/3} = r^{1/3}e^{i\theta/3}$. In the limit, on the upper path we have

$$z^{1/3} = x^{1/3}$$

and on the lower path we have

$$z^{1/3} = x^{1/3}e^{2\pi i/3} = x^{1/3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right).$$

Thus the integral over the whole contour converges to

$$\left(\frac{3}{2} - \frac{\sqrt{3}}{2}i\right) I.$$

There are simple poles at $\pm i$.

$$i^{1/3} = e^{i\pi/6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

and so the residue at i is

$$\frac{i^{1/3}}{i+i} = \frac{1}{2i}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right).$$

On the other hand

$$(-i)^{1/3} = e^{i\pi/2} = i$$

and so the residue at $-i$ is

$$\frac{(-i)^{1/3}}{-i-i} = \frac{-i}{2i}.$$

Therefore by the residue theorem we have

$$\left(\frac{3}{2} - \frac{\sqrt{3}}{2} i\right) I = \frac{2\pi i}{2i}\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i - i\right) = \pi\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right).$$

Thus

$$\int_0^\infty \frac{x^{1/3}\,\mathrm{d}x}{1+x^2} = \frac{\pi}{\sqrt{3}}.$$

(iii) Let's try to integrate by parts,

$$\int_0^\infty \log(1+x^2)\frac{\mathrm{d}x}{x^{1+\alpha}} = \frac{1}{\alpha}\int_0^\infty \frac{x^{1-\alpha}\mathrm{d}x}{(1+x^2)}.$$

There are two cases. If $\alpha = 1$ then consider the rational function

$$R(z) = \frac{1}{1+z^2}.$$

We integrate $R(z)$ around the standard contour from $-R$ to R along the real axis and along a semicircle of radius R in the upper half plane. As the radius R tends to infinity the integral along the semicircle goes to zero. If $R > 1$ then the contour goes around the only pole in the upper half plane which is at i.

The residue at i is

$$\frac{1}{i+i} = \frac{1}{2i}.$$

Taking the limit as R goes to infinity by the residue theorem we get

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = 2\pi i\left(\frac{1}{2i}\right) = \pi.$$

If $\alpha \neq 1$ the integral

$$I = \int_0^\infty \frac{x^{1-\alpha}\mathrm{d}x}{(1+x^2)}$$

is very similar the one in part (ii). Let's integrate over the same contour and use the same branch of the logarithm. The integral over the circle of radius R still goes to zero as R goes to infinity. The integral over the circle of radius ρ goes to zero as ρ goes to zero as the length of the path is proportional to ρ and the integrand is proportional to $\rho^{1-\alpha}$.

We define $z^{1-\alpha}$ using a branch of the logarithm which excludes the positive real axis and the argument θ satisfies $0 < \theta < 2\pi$. If $z = re^{i\theta}$ then $z^{1-\alpha} = r^{1-\alpha}e^{i(1-\alpha)\theta}$. In the limit, on the upper path we have

$$z^{1-\alpha} = x^{1-\alpha}$$

and on the lower path we have

$$z^{1-\alpha} = x^{1-\alpha}e^{2(1-\alpha)\pi i}.$$

Thus the integral over the whole contour converges to

$$(1 - e^{2(1-\alpha)\pi i})I.$$

There are simple poles at $\pm i$.

$$i^{1-\alpha} = e^{(1-\alpha)\pi i/2},$$

and so the residue at i is

$$\frac{i^{1-\alpha}}{i+i} = \frac{1}{2i} e^{(1-\alpha)\pi i/2}.$$

On the other hand

$$(-i)^{1-\alpha} = e^{(1-\alpha)3\pi i/2}.$$

and so the residue at $-i$ is

$$\frac{(-i)^{1-\alpha}}{-i-i} = \frac{-e^{(1-\alpha)3\pi i/2}}{2i}.$$

Therefore by the residue theorem we have

$$(1-e^{2(1-\alpha)\pi i})I = \frac{2\pi i}{2i}\left(e^{(1-\alpha)\pi i/2} - e^{(1-\alpha)3\pi i/2}\right) = \pi\left(e^{(1-\alpha)\pi i/2} - e^{(1-\alpha)3\pi i/2}\right).$$

Thus

$$\begin{aligned}\int_0^\infty \log(1+x^2)\frac{\mathrm{d}x}{x^{1+\alpha}} &= \frac{\pi}{\alpha}\frac{\left(e^{(1-\alpha)\pi i/2} - e^{-(1-\alpha)\pi i/2}\right)}{\left(e^{(1-\alpha)\pi i} - e^{-(1-\alpha)\pi i}\right)} \\ &= \frac{\pi}{\alpha}\frac{\sin(1-\alpha)\pi/2}{\sin(1-\alpha)\pi} \\ &= \frac{\pi}{2\alpha}\frac{1}{\cos(1-\alpha)\pi/2}.\end{aligned}$$

(iv)

$$\int_{|z|=\rho}\frac{|\mathrm{d}z|}{|z-a|^2} = \int_{|z|=\rho}\frac{i\rho\,\mathrm{d}z}{z(z-a)(\rho/z-\bar{a})} = \int_{|z|=\rho}\frac{-i\rho\,\mathrm{d}z}{(z-a)(\rho-\bar{a}z)}.$$

The rational function

$$\frac{-i\rho}{(z-a)(\rho-\bar{a}z)}$$

has poles at $z = a$ and $z = \rho/\bar{a}$. Only one is inside the circle of radius ρ. The residue as a is

$$\frac{-i\rho}{\rho-|a|^2}$$

and the residue as $\rho/\bar{a}$ is

$$\frac{-i\rho}{\rho/\bar{a}-a} = \frac{-i\bar{a}\rho}{\rho-|a|^2}.$$

By the residue theorem

$$\int_{|z|=\rho} \frac{|\mathrm{d}z|}{|z-a|^2} = \begin{cases} \frac{2\pi\rho}{\rho-|a|^2} & \text{if } |a| < \rho \\ \frac{2\pi\bar{a}\rho}{\rho-|a|^2} & \text{if } |a| > \rho. \end{cases}$$