MODEL ANSWERS TO THE FIFTH HOMEWORK

1. The product of Hausdorff is Hausdorff so that $X \times Y$ is Hausdorff. If U_i and V_j are countable bases for X and Y then $U_i \times V_j$ is a countable base for $X \times Y$; thus $X \times Y$ is 2nd countable.
If $h_i \colon U_i \longrightarrow V_i$ is a chart on X_i, $i = 1$ and 2 then

$$h_1 \times h_2 \colon U_1 \times U_2 \longrightarrow V_1 \times V_2$$

is a homeomorphism and $V_1 \times V_2 \subset \mathbb{R}^{m+n}$ is an open subset. Thus $X \times Y$ is a topological manifold of dimension $m + n$.
2. For every point $z_0 \in U$, pick a disk of radius ρ centred at z_0, contained in U. Then Harnack's inequality states that

$$\frac{\rho - r}{\rho + r} u(z_0) \leq u(z) \leq \frac{\rho + r}{\rho - r} u(z_0),$$

for every $r = |z - z_0| < \rho$. If we take $r \leq \rho/2$, then we get

$$\frac{1}{3} u(z_0) \leq u(z) \leq 3u(z_0).$$

By compactness we may find finitely many points $z_1, z_2, \dots, z_k \in E$ and circles of radius $\rho_1, \rho_2, \dots, \rho_k$ centred at these points contained in U, such that the disc $|z - z_i| < \rho/2$ cover U and

$$\frac{1}{3} u(z_i) \leq u(z) \leq 3u(z_i).$$

for every point $|z - z_i| < \rho_i/2$. Let $M = 3^k$. Suppose that z and z' are two points of E. Possibly relabelling, we may assume that

$$\begin{aligned} |z - z_1| &< \rho_1/2 \\ |z_{i+1} - z_i| &< \rho_i/2 \\ |z' - z_l| &< \rho_l/2. \end{aligned}$$

But then

$$\begin{aligned} u(z_1) &< 3u(z) \\ u(z_i) &< 3u(z_{i+1}) \\ u(z') &< 3u(z_l), \end{aligned}$$

so that

$$u(z') < 3^l u(z) \leq M u(z).$$

3. x is the real part of a holomorphic function (namely the identity), so that both x and $-x$ are harmonic. But then

$$|x| = \max(x, -x),$$

is subharmonic.
Now

$$\frac{1}{2\pi}\int_0^{2\pi} r^\alpha \, d\theta \geq 0.$$

for any circle of radius r centred at the origin. Thus it suffices to show that the function $|z|^\alpha$ is subharmonic outside the origin. But $z \longrightarrow z^\alpha$ is locally a holomorphic function away from the origin. As $|z|^\alpha = |z^\alpha|$, it suffices to check that r^2 is subharmonic, which is easy as,

$$\Delta r^2 = 2 > 0.$$

Now

$$\frac{\partial \log(1 + r^2)}{\partial x} = \frac{2x}{1 + r^2}.$$

Thus

$$\Delta \log(1 + r^2) = \frac{4(1 + r^2) - 2x^2 - 2y^2}{(1 + r^2)^2} = \frac{4 + 2r^2}{(1 + r^2)^2} > 0.$$

Thus $\log(1 + |z|^2)$ is subharmonic.
4. Let M be the supremum of v. Then we can find points $z_1, z_2, \dots$ such that

$$\lim_{i\to\infty} v(z_i) = M.$$

As E is compact, possibly passing to a subsequence, we may assume that the points $z_1, z_2, \dots$ converge to a point $z_0 \in E$. But then

$$\begin{aligned} M &= \lim_{i\to\infty} v(z_i) \\ &\leq \limsup_{i\to\infty} v(z_i) \\ &\leq v(z_0) \leq M, \end{aligned}$$

where the penultimate inequality follows as f is upper semi-continuous. But then $v(z_0) = M$, as required.
5.

Theorem 0.1. *Let $U \subset \mathbb{C}$ be a region. Suppose that the subharmonic functions $v_1, v_2, \dots$ converge uniformly on compact subsets to the function v.*
Then v is subharmonic.

Proof. Since we have uniform convergence on compact subsets, v is continuous. Let $z_0 \in U$. Pick a circle about z_0 of sufficiently small radius r so that it is contained in U. Then

$$v_i(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v_i(re^{i\theta}) \, d\theta.$$

Taking the limit of both sides, and using the fact that we have uniform convergence on the circle, since it is compact, we have

$$v(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta}) \, d\theta.$$

But then v is subharmonic. □