PRACTICE PROBLEMS FOR THE FINAL

Here are some practice problems for the final. They contain bookwork plus some problems culled from various locations.

1. Let $f\colon \Delta \longrightarrow \Delta$ be a holomorphic map of the disc to itself. If $0 < |f(z)| < 1$ for all $z \in \Delta$ show that for any $z \in \Delta$

$$|f(z)| \leq |f(0)|^{\frac{1-|z|}{1+|z|}}.$$

2. Give an example a harmonic function on the unit disc Δ that cannot be represented as the difference of two non-negative harmonic functions on Δ.

3. Let $f_1, f_2, \dots$ be a sequence of holomorphic functions on a region U. Suppose that $f_n(a)$ converges for some $a \in U$ and that $\operatorname{Re} f_1, \operatorname{Re} f_2, \dots$ is a normal family on U. Prove that $f_1, f_2, \dots$ is a normal family.

4. Let $u_1, u_2, \dots$ be a sequence of harmonic functions in a region U such that $|u_n(z)| < C$ on U for some C and all n. Suppose that there is an open subset $V \subset U$ so that $u_n(z)$ converges for any $z \in V$. Prove that the limit converges for all $z \in U$ and that the limit is harmonic.

5. Let u be a harmonic function on a region U. Show that $u(U)$ is open in $\mathbb{R}$.

6. Let $f_1, f_2, \dots$ be the iterates of the function $\sin z$, so that $f_1(z) = \sin z$ and $f_{n+1}(z) = f_n(\sin z)$. Show that $f_1, f_2, \dots$ is not locally bounded at the origin.

7. Let $u(z)$ be a bounded harmonic function on the unit disc Δ. Suppose that

$$\lim_{r\to 1} u(re^{i\theta}) = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi, \end{cases}$$

(where r approaches 1 from below). Find $u(1/2)$.

8. Let $\mathfrak{F}$ be a family of holomorphic functions in a neighbourhood of the closed unit disc $\bar{\Delta}$. If

$$\int_0^{2\pi} |f(e^{i\theta})|^{1/2}\, \mathrm{d}\theta \leq 1$$

for all $f \in \mathfrak{F}$. Prove that $\mathfrak{F}$ is a normal family on the open unit disc Δ.