6. Quotient groups II

Now suppose that we drop the hypothesis that H is the kernel of a homomorphism and replace this with the hypothesis that H is normal. Then we still get a group. The key thing to check is:

Theorem 6.1. *Let H be a subgroup of a group G.*

Then the following rule for multiplication of left cosets

$$(aH)(bH) = abH$$

if and only if H is normal in G.

Proof. Suppose that this rule of multiplication is well-defined. Pick $a \in G$. We have to show that

$$aH = Ha.$$

Pick $x \in aH$. There are two possible ways to define the product

$$aHa^{-1}H.$$

The really obvious way is to say

$$aHa^{-1}H = aa^{-1}H = eH = H.$$

As $aH = xH$ another way is to say

$$aHa^{-1}H = xHa^{-1}H = xa^{-1}H.$$

If the rule of multiplication is well-defined we must have

$$xa^{-1}H = H.$$

In this case $xa^{-1} \in H$, so that there is an element $h \in H$ such that $xa^{-1} = h$. Multiplying both sides of the right by a we get

$$x = ha$$

so that $x \in Ha$. Thus the LHS is a subset of the RHS. By symmetry the RHS is a subset of the LHS. But then $aH = Ha$. As a is arbitrary, H is normal.

We have already proved that if H is normal then multiplication of left cosets is well-defined but let's prove it again.

Suppose that H is normal. If $a'H = aH$ and $b'H = bH$ then $a' = ah_1$ and $b' = bh_2$, for some h_1 and h_2. On the other hand, we may find $h_3 \in H$ so that

$$h_1 b = bh_3 \qquad \text{since} \qquad bH = Hb.$$

It follows that

$$\begin{aligned} a'b' &= (ah_1)(bh_2) \\ &= a(h_1b)h_2 \\ &= a(bh_3)h_2 \\ &= (ab)(h_3h_2). \end{aligned}$$

Thus

$$a'b'H = abH,$$

and the multiplication is indeed well-defined. □

Corollary 6.2. *Let H be a normal subgroup of a group G.*

Then the left cosets of H in G form a group with multiplication defined by

$$(aH)(bH) = abH.$$

Proof. We have already checked that this rule of multiplication is well-defined in (6.1). We first check associativity. Suppose that aH, bH and cH are three left cosets. Then

$$\begin{aligned} aH(bHcH) &= aH(bcH) \\ &= a(bc)H \\ &= (ab)cH \\ &= (abH)cH \\ &= (aHbH)cH. \end{aligned}$$

Thus we have associativity.

We claim that $eH = H$ is the identity. Suppose that aH is another left coset. We have

$$aHeH = aeH = aH = eaH = eHaH.$$

Thus eH is the identity.

Finally we check that $a^{-1}H$ is the inverse of aH. We have

$$aHa^{-1}H = aa^{-1}H = eH = a^{-1}aH = a^{-1}HaH.$$

Thus $a^{-1}H$ is the inverse of aH. □

Definition 6.3. *The group of left cosets with multiplication defined in* (6.1) *is called the* ***quotient group****, denoted G/H.*

Example 6.4. *If $n\mathbb{Z} = \langle n \rangle$ is the subgroup of $\mathbb{Z}$ of all multiples of n then $\langle n \rangle$ is a normal subgroup.*

Indeed it is the kernel of the map

$$\gamma\colon \mathbb{Z} \longrightarrow \mathbb{Z}_n,$$

which sends a number to its remainder. The quotient group

$$\mathbb{Z}/\langle n\rangle$$

is isomorphic to $\mathbb{Z}_n$ the integers modulo n.

Example 6.5. *Let $A_3 \subset S_3$ be the alternating subgroup.*

Then A_3 is the kernel of the homomorphism $\phi\colon S_3 \longrightarrow \mathbb{Z}_2$ and so A_3 is normal. Let's consider the group table of $G = S_3$.

$*$	e	$(1,2,3)$	$(1,3,2)$	$(2,3)$	$(1,3)$	$(1,2)$
e	e	$(1,2,3)$	$(1,3,2)$	$(2,3)$	$(1,3)$	$(1,2)$
$(1,2,3)$	$(1,2,3)$	$(1,3,2)$	e	$(1,2)$	$(2,3)$	$(1,3)$
$(1,3,2)$	$(1,3,2)$	e	$(1,2,3)$	$(1,3)$	$(1,2)$	$(2,3)$
$(2,3)$	$(2,3)$	$(1,3)$	$(1,2)$	e	$(1,2,3)$	$(1,3,2)$
$(1,3)$	$(1,3)$	$(1,2)$	$(2,3)$	$(1,3,2)$	e	$(1,2,3)$
$(1,2)$	$(1,2)$	$(2,3)$	$(1,3)$	$(1,2,3)$	$(1,3,2)$	e

Note that things have been arranged so that the first three entries in each row (and so column) are the elements of A_3. The index of A_3 in S_3 is $2 = 6/3$. So there are two left cosets, the elements of A_3 and everything else. If you shade the elements of A_3 one colour, say red, and everything else another colour, say blue, then we get the quotient group:

$*$	R	B
R	R	B
B	B	$R.$

Obviously this group is isomorphic to $\mathbb{Z}_2$.

On the other hand, the same thing won't work if we start with the subgroup $H = \{e, (1,2)\}$. In this case there are three left cosets,

$$H = \{e, (1,2)\} \quad (1,2,3)H = \{(1,2,3), (1,3)\} \quad \text{and} \quad (1,3,2)H = \{(1,3,2), (2,3))\}.$$

Imagine these three cosets are represented by three colours, red, blue and yellow. How should we define the product of a red times a blue? Well, take a red element and a blue element and multiply them together. We could take e for a red and $(1,2,3)$ for a blue. Then $e(1,2,3) = (1,2,3)$, which is blue. So, from this point of view, we should say red times blue is blue. But now suppose we take $(1,2)$ and $(1,3)$ another red and blue. Then $(1,2)(1,3) = (1,3,2)$, a yellow. So, from this point of view, we should say red times blue is yellow. So there is no consistent way to multiply red times blue.