8. Simple groups

Proposition 8.1. *Let G be a group and let H be a subgroup of index 2.*

Then H is normal in G.

Proof. What are the left cosets of H? One of them is $eH = H$. There are only two left cosets, by Lagrange, and so the other one is everything else.

What are the right cosets of H? One of them is $He = H$. There are only two right cosets, by Lagrange, and so the other one is everything else.

Therefore the left cosets and right cosets are the same and so H is normal. □

Proposition 8.2. *A_4 contains no subgroups of order 6.*

Proof. Suppose not, let H be a subgroup of order 6. Then the index of H is

$$\frac{|A_4|}{H} = \frac{12}{6} = 2.$$

Thus H is normal by (8.1).

The quotient group has two elements, the identity eH and the other element. The square of every element in the quotient group is the identity. It follows that if $\sigma \in A_4$ then

$$(\sigma H)^2 = \sigma^2 H = H,$$

or what comes to the same thing $\sigma^2 \in H$.

On the other hand,

$$(1,2,3)^2 = (1,2,3)(1,2,3) = (1,3,2).$$

Since $(1,3,2)$ is a square, $(1,3,2) \in H$. By symmetry every three cycle is a square and so every three cycle must belong to H. There are

$$\frac{4 \cdot 3 \cdot 2}{3} = 8,$$

three cycles and so H would contain at least 8 elements, a contradiction. □

Proposition 8.3. *Let G be a cyclic group and let N be a subgroup.*

Then G/N is a cyclic group.

Proof. Suppose that a is a generator of G. Consider the left coset aN. We check that this is a generator of G/N. Observe that

$$(aN)^k = a^k N \qquad \text{for every} \qquad k \in \mathbb{Z}.$$

As we go through the powers of a we get every element of G. Therefore we surely get every left coset. But then aN is a generator of G/N. $\square$

Note that if G is a group and N is a normal subgroup then we can think of G as being built from the subgroup N and the quotient group G/N. Thinking in these terms the building blocks of groups are groups without interesting normal subgroups:

Definition 8.4. *We say a group G is* ***simple*** *if G contains no non-trivial normal subgroups, that is, if N is a normal subgroup of G then either $N = G$ or $N = \{e\}$.*

We already know two interesting infinite sequences of simple groups:

$$\mathbb{Z}_p$$

is a simple group if and only if p is a prime number; indeed if p is prime then Lagrange says that there are no non-trivial subgroups at all. On the other hand, if $p = ab$ then $\langle a \rangle$ is a subgroup of order b which is automatically normal, as $\mathbb{Z}_p$ is abelian.

Another infinite sequence of groups is given by S_n is groups. However A_n is always a normal subgroup, since A_n has index 2. On the other hand, A_n gives an infinite sequence of simple groups:

Theorem 8.5. *A_n is simple if and only if $n \neq 4$.*

All finite simple groups have been classified. The classification took over a century of hard work. There are 17 infinite sequences of simple groups. More interestingly there are 26 sporadic groups, which don't fit into any infinite sequence of simple groups.

The monster group is the largest sporadic group; it has order

$$808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000.$$

The prime factorisation of this number is

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

It is a subgoup of the group of linear symmetries of a real vector space of dimension $196883 = 47 \cdot 59 \cdot 71$,

$$\mathbb{R}^{196883},$$

in other words it is a subgroup of

$$\mathrm{GL}(196883, \mathbb{R}),$$

the group of 196883×196883 invertible matrices.

All finite groups appear as subgroups of the permutation group. The smallest n such that the monster is a subgroup of S_n is

$$2^4 \cdot 3^7 \cdot 5^3 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 29 \cdot 41 \cdot 59 \cdot 71.$$