We introduce the main object of study for 103B.

**Definition 9.1.** A **ring** is a set $R$, together with two binary operations addition and multiplication, denoted $+$ and $\cdot$ respectively, which satisfy the following axioms. Firstly $R$ is an abelian group under addition, with zero as the identity.

1. **(Associativity)** For all $a$, $b$ and $c$ in $R$,
   \[(a + b) + c = a + (b + c).\]

2. **(Zero)** There is an element $0 \in R$ such that for all $a$ in $R$,
   \[a + 0 = 0 + a.\]

3. **(Additive Inverse)** For all $a$ in $R$, there exists $b \in R$ such that
   \[a + b = b + a = 0.\]
   $b$ will be denoted $-a$.

4. **(Commutativity)** For all $a$ and $b$ in $R$,
   \[a + b = b + a.\]

Secondly multiplication is also associative.

5. **(Associativity)** For all $a$, $b$ and $c$ in $R$,
   \[(a \cdot b) \cdot c = a \cdot (b \cdot c).\]

Finally we require that addition and multiplication are compatible in an obvious sense.

6. **(Distributivity)** For all $a$, $b$ and $c$ in $R$, we have
   \[a \cdot (b + c) = a \cdot b + a \cdot c,\]
   \[(b + c) \cdot a = b \cdot a + c \cdot a.\]

**Example 9.2.** The complex numbers $\mathbb{C}$ are a ring.

**Definition 9.3.** Let $R$ be a ring and let $S$ be a subset. We say that $S$ is a **subring** of $R$, if $S$ becomes a ring, with the induced addition and multiplication.

**Lemma 9.4.** Let $R$ be a ring and let $S$ be a non-empty subset.

Then $S$ is a subring if and only if $S$ is closed under addition, additive inverses and multiplication.

**Proof.** Similar proof as for groups.
Example 9.5. The following tower of subsets
\[ \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \]
is in fact a tower of subrings. Thus the integers, rational numbers, reals and complex numbers are all rings.

Example 9.6. Let \( \mathbb{Z}_n \) denote the integers modulo \( n \). We already know that this forms a group under addition. It is also the case that multiplication is well-defined so that \( \mathbb{Z}_n \) is a ring.

It is interesting to see what happens in a specific example. Suppose that \( n = 6 \). In this case \( 0 = [0] \). However note that one curious feature is that
\[ [2][3] = [2 \cdot 3] = [6] = [0], \]
so that the product of two non-zero elements of \( R \) might in fact be zero.

Definition-Lemma 9.7. Let \( R \) be a ring and let \( n \) be a positive integer. \( M_n(R) \) denotes the set of all \( n \times n \) matrices with entries in \( R \). Given two such matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \), we define \( A + B \) as \((a_{ij} + b_{ij})\). The product of \( A \) and \( B \) is also defined in the usual way. That is, the \( ij \) entry of \( AB \) is the dot product of the \( i \)th row of \( A \) and the \( j \)th column of \( B \).

With this rule of addition and multiplication \( M_n(R) \) becomes a ring, with zero given as the zero matrix (every entry equal to zero).

Proof. This is standard but somewhat tedious to check. \( \square \)

Note that if \( n = 1 \) then \( M_1(R) \) is simply a copy of \( R \).
To fix ideas, let us consider an easy example of a matrix ring.

Example 9.8. Let \( R = \mathbb{Z}_6 \) be the ring of integers modulo 6 and take \( n = 2 \). Take
\[ A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 5 \\ 1 & 2 \end{pmatrix} \]
Then
\[ AB = \begin{pmatrix} 3 + 1 & 15 + 2 \\ 2 + 4 & 10 + 8 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 6 & 0 \end{pmatrix}. \]

Definition-Lemma 9.9. Consider the set \( F \) of functions from \( \mathbb{R} \) to \( \mathbb{R} \). We already know how to add to such functions: given \( f \) and \( g \in F \), define \( f + g \) by the rule,
\[ (f + g)(x) = f(x) + g(x) \in \mathbb{R}, \]
where \( x \in \mathbb{R} \) and addition is in \( \mathbb{R} \). In words, we add the functions pointwise.
Under addition $F$ becomes a group. There is an obvious way to multiply two functions: define $f \cdot g$ by the rule,

$$(f \cdot g)(x) = f(x) \cdot g(x) \in \mathbb{R},$$

where $x \in \mathbb{R}$ and multiplication is in $\mathbb{R}$. In words we multiply two functions pointwise.

With this rule of addition and multiplication $F$ becomes a ring.

**Proof.** Again, all of this is easy to check. We check associativity of addition and leave the rest to the reader. Suppose that $f$, $g$ and $h$ are three functions from $X$ to $R$. We want to prove

$$(f + g) + h = f + (g + h).$$

Since both sides are functions from $\mathbb{R}$ to $\mathbb{R}$, it suffices to prove that they have the same effect on any real number $x \in \mathbb{R}$.

$$
((f + g) + h)(x) = (f + g)(x) + h(x)
= (f(x) + g(x)) + h(x)
= f(x) + (g(x) + h(x))
= f(x) + (g + h)(x)
= (f + (g + h))(x).
$$

$\square$

**Example 9.10.** $n\mathbb{Z}$ is a subring of $\mathbb{Z}$, since the sum and product of two multiples of $n$ is a multiple of $n$.

**Example 9.11.** If $R$ and $S$ are rings then the Cartesian product is naturally a ring. We have already seen that $R \times S$ is naturally an abelian group under addition. If we define multiplication by the rule:

$$(r_1, s_1) \cdot (r_2, s_2) = (r_1r_2, s_1s_2)$$

then it is not hard to see that $R \times S$ is a ring.