

### 3. A LITTLE CATEGORY THEORY

**Definition 3.1.** A **category**  $\mathcal{C}$  consists of two things. The first is the **objects** of the category. The second is, given any two objects,  $X$  and  $Y$ , we associate a collection of **morphisms**, denoted  $\text{Hom}(X, Y)$ . Given three objects,  $X, Y$  and  $Z$  and two morphisms  $f \in \text{Hom}(X, Y)$ ,  $g \in \text{Hom}(Y, Z)$ , there is a rule of composition,

$$g \circ f \in \text{Hom}(X, Z).$$

A category satisfies the following two axioms.

- (1) *Composition is associative.* That is, given  $f \in \text{Hom}(X, Y)$ ,  $g \in \text{Hom}(Y, Z)$ ,  $h \in \text{Hom}(Z, W)$ , we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (2) *There is an element  $I_X \in \text{Hom}(X, X)$  that acts as an identity, so that if  $f \in \text{Hom}(X, Y)$  and  $g \in \text{Hom}(Y, X)$  we have,*

$$f \circ I_X = f,$$

and

$$I_X \circ g = g.$$

Given  $f \in \text{Hom}(X, Y)$  we say that  $g \in \text{Hom}(Y, X)$  is the inverse of  $f$  if  $f \circ g = I_Y$  and  $g \circ f = I_X$ . In this case we say that  $f$  is an **isomorphism** and we say that  $X$  and  $Y$  are **isomorphic**.

Given most advanced mathematics classes, there is normally a naturally associated category.

One natural category is the category of groups. The objects are the collection of all groups and the morphisms are homomorphisms. Composition of morphisms is composition of functions. It is not hard to see that all the axioms are satisfied. In fact the only thing we have not checked is that composition of homomorphisms is a homomorphism, which is an exercise for the reader.

Two groups are isomorphic if and only if they are isomorphic as objects of the category.

Another very natural category is the category of sets. Here a morphism is any function. In fact the category of groups is a sub category of the category of sets (in a sense which can be made precise).

Yet another category is the category of topological spaces. The objects are topological spaces and the morphisms are continuous maps. Two topological spaces are then homeomorphic if and only if they are isomorphic as objects of this category. The category of metric spaces is a subcategory of the category of topological spaces.

The category of vector spaces, is the category whose objects are vector spaces and whose morphisms are linear maps.

Note that all of these categories are subcategories of the category of sets, in an obvious way.

**Definition 3.2.** *Let*

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow b & & \downarrow c \\ C & \xrightarrow{d} & D \end{array}$$

*be a collection of objects and morphisms belonging to a category.*

*We say that the diagram **commutes** if the two morphisms from  $A$  to  $D$  are the same, that is*

$$c \circ a = d \circ b.$$

In a category, the focus of interest is not the objects, but the morphisms between the objects. In this sense, we would like to characterise the notion of the quotient group in a way that does not make explicit reference to the elements of  $G/H$ , but rather define everything in terms of homomorphisms between groups. Even though this is somewhat abstract, there is an obvious advantage to this approach; as a set  $G/H$  is rather complicated, its elements are left cosets, which are themselves sets. But really we only need to know what  $G/H$  is up to isomorphism.

**Definition 3.3.** *Let  $G$  be a group and let  $H$  be a normal subgroup.*

*The **categorical quotient** of  $G$  by  $H$  is a group  $Q$  together with a homomorphism  $u: G \rightarrow Q$ , such that*

- (1)  *$u$  is onto.*
- (2) *The kernel of  $u$  is  $H$ .*

*which is universal amongst all such homomorphisms in the following sense.*

*Suppose that  $\phi: G \rightarrow G'$  is any homomorphism such that the kernel of  $\phi$  contains  $H$ . Then there is a unique induced homomorphism*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ \downarrow u & \searrow \text{dotted} & \\ Q & & \end{array}$$

*which makes the diagram commute.*

Note that in the definition of the categorical quotient, the most important part of the definition refers to the homomorphism  $u$ , and the universal property that it satisfies.

**Theorem 3.4.** *The category of groups admits categorical quotients.*

*That is, given a group  $G$  and a normal subgroup  $H$ , there is a categorical quotient group  $Q$ . Furthermore,  $Q$  is unique, up to a unique isomorphism.*

*Proof.* We first prove existence. Let  $G/H$  be the quotient group and let  $u: G \rightarrow G/H$  be the natural homomorphism. I claim that this pair forms a categorical quotient. Thus we have to prove that  $u$  is universal.

To this end, suppose that we are given a homomorphism  $\phi: G \rightarrow G'$  whose kernel contains  $H$ . Define a map

$$f: G/H \rightarrow G'$$

by sending  $gH$  to  $\phi(g)$ . It is clear that the condition that the diagram commutes, forces this definition of  $f$ . We have to check that  $f$  is well-defined.

Suppose that  $g_1H = g_2H$ . We need to check that  $\phi(g_1) = \phi(g_2)$ . As  $g_1H = g_2H$ , it follows that  $g_2 = g_1h$ , for some  $h \in H$ . In this case

$$\begin{aligned} \phi(g_2) &= \phi(g_1h) \\ &= \phi(g_1)\phi(h) \\ &= \phi(g_1), \end{aligned}$$

where we use the fact that  $h$  is in the kernel of  $\phi$ . Thus the map  $f$  is well-defined and  $f$  is unique.

Now we check that  $f$  is a homomorphism. Suppose that  $x$  and  $y$  are in  $G/H$ . Then  $x = g_1H$  and  $y = g_2H$ , for some  $g_1$  and  $g_2$  in  $G$ . In this case

$$\begin{aligned} f(xy) &= f(g_1g_2H) \\ &= \phi(g_1g_2) \\ &= \phi(g_1)\phi(g_2) \\ &= f(g_1H)f(g_2H) = f(x)f(y). \end{aligned}$$

Thus  $f$  is a homomorphism. Finally we check that the diagram above commutes. Suppose that  $g \in G$ . Going along the top we get  $\phi(g)$ . Going down first, we get  $gH$  and then going diagonally we get  $f(gH) = \phi(g)$ , by definition of  $f$ .

Thus  $G/H$  is a categorical quotient. In particular categorical quotients exist.

Now we prove that categorical quotients are unique, up to unique isomorphism. Suppose that  $Q_1$  and  $Q_2$  are two such categorical quotients. As  $Q_1$  is a categorical quotient and there is a homomorphism  $u_2: G \rightarrow Q_2$  it follows that there is an induced homomorphism  $f: Q_1 \rightarrow Q_2$ , which makes the following diagram commute

$$\begin{array}{ccc} G & \xrightarrow{u_2} & Q_2 \\ u_1 \downarrow & \nearrow f & \\ Q_1 & & \end{array}$$

By symmetry there is a homomorphism  $g: Q_2 \rightarrow Q_1$ , which makes the same diagram commute, with 1 and 2 switched. Consider the composition  $f \circ g: Q_2 \rightarrow Q_2$ . This is a homomorphism which make the following diagram commute

$$\begin{array}{ccc} G & \xrightarrow{u_2} & Q_2 \\ u_2 \downarrow & \nearrow & \\ Q_2 & & \end{array}$$

However there is one homomorphism that makes the diagram commute, namely the identity. By uniqueness,  $f \circ g$  must be the identity. Similarly  $g \circ f$  must be the identity. So  $f$  and  $g$  are inverses of each other, and hence isomorphisms. Note that  $f$  itself is unique, since its existence was given to us by the universal property of  $u_1$ .

Thus the quotient is unique, up to a unique isomorphism. □

Now it is easy to use this to deduce the isomorphism Theorems.

We have already seen that given any group  $G$  and a normal subgroup  $H$ , there is a natural homomorphism  $\phi: G \rightarrow G/H$ , whose kernel is  $H$ . In fact we will see that this map is not only natural, it is in some sense the only such map.

**Theorem 3.5** (First Isomorphism Theorem). *Let  $\phi: G \rightarrow G'$  be a homomorphism of groups. Suppose that  $\phi$  is onto and let  $H$  be the kernel of  $\phi$ .*

*Then  $G'$  is isomorphic to  $G/H$ .*

*Proof.* By the universal property of a quotient, there is a natural homomorphism

$$f: G/H \rightarrow G'.$$

As  $f$  makes the following diagram commute,

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ u \downarrow & \nearrow f & \\ G/K & & \end{array}$$

it follows that  $f$  is surjective. It remains to prove that  $f$  is injective. Suppose that  $x$  is in the kernel of  $f$ . Then  $x$  has the form  $gH$  and by definition of  $f$ ,  $f(x) = \phi(g)$ . Thus  $g$  is in the kernel of  $\phi$  and so  $g \in H$ . In this case  $x = H$ , the identity of  $G/H$ . So the kernel of  $f$  is trivial and  $f$  is injective. Hence  $f$  is an isomorphism.  $\square$

**Definition 3.6.** Let  $G$  be a group and let  $H$  and  $K$  be two subgroups of  $G$ .

$H \vee K$  denotes the subgroup generated by the union of  $H$  and  $K$ .

In general, it is hard to identify  $H \vee K$  as a set. However,

**Theorem 3.7** (Second Isomorphism Theorem). Let  $G$  be a group, let  $H$  be a subgroup and let  $N$  be a normal subgroup. Then

$$H \vee N = HN = \{ hn \mid h \in H, n \in N \}.$$

Furthermore  $H \cap N$  is a normal subgroup of  $H$  and the two groups  $H/H \cap N$  and  $HN/N$  are isomorphic.

*Proof.* The pairwise products of the elements of  $H$  and  $N$  are certainly elements of  $HN$ . Thus the RHS of the equality above is a subset of the LHS. The RHS is clearly non-empty and so it suffices to prove that the RHS is closed under products and inverses.

Suppose that  $x$  and  $y$  are elements of the RHS. Then  $x = h_1n_1$  and  $y = h_2n_2$ , where  $h_i \in H$  and  $n_i \in N$ . Now  $h_2^{-1}n_1h_2 = n_3 \in N$ , as  $N$  is normal in  $G$ . So  $n_1h_2 = h_2n_3$ . In this case

$$\begin{aligned} xy &= (h_1n_1)(h_2n_2) \\ &= (h_1(n_1h_2))n_2 \\ &= (h_1(h_2n_3))n_2 \\ &= (h_1h_2)(n_3n_2), \end{aligned}$$

which shows that  $xy$  has the correct form. On the other hand, suppose  $x = hn$ . Then  $hnh^{-1} = m \in N$  as  $N$  is normal and so  $hn^{-1}h^{-1} = m^{-1}$ . In this case

$$\begin{aligned} x^{-1} &= n^{-1}h^{-1} \\ &= hm^{-1}, \end{aligned}$$

so that  $x^{-1}$  is of the correct form.

Hence the first statement. Let  $H \rightarrow HN$  be the natural inclusion. As  $N$  is normal in  $G$ , it is certainly normal in  $HN$ , so that we may compose the inclusion with the natural projection map to get a homomorphism

$$H \rightarrow HN/N$$

This map sends  $h$  to  $hN$ .

Suppose that  $x \in HN/N$ . Then  $x = hnN = hN$ , where  $h \in H$ . Thus the homomorphism above is clearly surjective. Suppose that  $h \in H$  belongs to the kernel. Then  $hN = N$  the identity coset, so that  $h \in N$ . Thus  $h \in H \cap N$ . The result then follows by the First Isomorphism Theorem applied to the map above.  $\square$

It is easy to prove the Third isomorphism Theorem from the First.

**Theorem 3.8** (Third Isomorphism Theorem). *Let  $K \subset H$  be two normal subgroups of a group  $G$ .*

*Then*

$$G/H \simeq (G/K)/(H/K).$$

*Proof.* Consider the natural map  $G \rightarrow G/H$ . The kernel,  $H$ , contains  $K$ . Thus, by the universal property of  $G/K$ , it follows that there is a homomorphism  $G/K \rightarrow G/H$ .

This map is clearly surjective. In fact, it sends the left coset  $gK$  to the left coset  $gH$ . Now suppose that  $gK$  is in the kernel. Then the left coset  $gH$  is the identity coset, that is  $gH = H$ , so that  $g \in H$ . Thus the kernel consists of those left cosets of the form  $gK$ , for  $g \in H$ , that is  $H/K$ . The result now follows by the first Isomorphism Theorem.  $\square$