

## 5. FIELD OF FRACTIONS

The rational numbers  $\mathbb{Q}$  are constructed from the integers  $\mathbb{Z}$  by adding inverses. In fact a rational number is of the form  $a/b$ , where  $a$  and  $b$  are integers. Note that a rational number does not have a unique representative in this way. In fact

$$\frac{a}{b} = \frac{ka}{kb}.$$

So really a rational number is an equivalence class of pairs  $[a, b]$ , where two such pairs  $[a, b]$  and  $[c, d]$  are equivalent if and only if  $ad = bc$ .

Now given an arbitrary integral domain  $R$ , we can perform the same operation.

**Definition-Lemma 5.1.** *Let  $R$  be any integral domain. Let  $N$  be the subset of  $R \times R$  such that the second coordinate is non-zero.*

*Define an equivalence relation  $\sim$  on  $N$  as follows.*

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

*Proof.* We have to check three things, reflexivity, symmetry and transitivity.

Suppose that  $(a, b) \in N$ . Then

$$a \cdot b = a \cdot b$$

so that  $(a, b) \sim (a, b)$ . Hence  $\sim$  is reflexive.

Now suppose that  $(a, b), (c, d) \in N$  and that  $(a, b) \sim (c, d)$ . Then  $ad = bc$ . But then  $cb = da$ , as  $R$  is commutative and so  $(c, d) \sim (a, b)$ . Hence  $\sim$  is symmetric.

Finally suppose that  $(a, b), (c, d)$  and  $(e, f) \in R$  and that  $(a, b) \sim (c, d), (c, d) \sim (e, f)$ . Then  $ad = bc$  and  $cf = de$ . Then

$$\begin{aligned} (af)d &= (ad)f \\ &= (bc)f \\ &= b(cf) \\ &= (be)d. \end{aligned}$$

As  $(c, d) \in N$ , we have  $d \neq 0$ . Cancelling  $d$ , we get  $af = be$ . Thus  $(a, b) \sim (e, f)$ . Hence  $\sim$  is transitive.  $\square$

**Definition-Lemma 5.2.** *The **field of fractions of  $R$** , denoted  $F$  is the set of equivalence classes, under the equivalence relation defined above. Given two elements  $[a, b]$  and  $[c, d]$  define*

$$[a, b] + [c, d] = [ad + bc, bd] \quad \text{and} \quad [a, b] \cdot [c, d] = [ac, bd].$$

With these rules of addition and multiplication  $F$  becomes a field. Moreover there is a natural injective ring homomorphism

$$\phi: R \longrightarrow F,$$

so that we may identify  $R$  as a subring of  $F$ . In fact  $\phi$  is universal among all such injective ring homomorphisms whose targets are fields.

*Proof.* First we have to check that this rule of addition and multiplication is well-defined. Suppose that  $[a, b] = [a', b']$  and  $[c, d] = [c', d']$ . By commutativity and an obvious induction (involving at most two steps, the only real advantage of which is to simplify the notation) we may assume  $c = c'$  and  $d = d'$ . As  $[a, b] = [a', b']$  we have  $ab' = a'b$ . Thus

$$\begin{aligned} (a'd + b'c)(bd) &= a'bd^2 + bb'cd \\ &= ab'd^2 + bb'cd \\ &= (ad + bc)(b'd). \end{aligned}$$

Thus  $[a'd + b'c, b'd] = [ad + bc, bd]$ . Thus the given rule of addition is well-defined. It can be shown similarly (and in fact more easily) that the given rule for multiplication is also well-defined.

We leave it is an exercise for the reader to check that  $F$  is a ring under addition and that multiplication is associative. For example, note that  $[0, 1]$  plays the role of 0 and  $[1, 1]$  plays the role of 1.

Given an element  $[a, b]$  in  $F$ , where  $a \neq 0$ , then it is easy to see that  $[b, a]$  is the inverse of  $[a, b]$ . It follows that  $F$  is a field.

Define a map

$$\phi: R \longrightarrow F,$$

by the rule

$$\phi(a) = [a, 1].$$

Again it is easy to check that  $\phi$  is indeed an injective ring homomorphism and that it satisfies the given universal property.  $\square$

**Example 5.3.** If we take  $R = \mathbb{Z}$ , then of course the field of fractions is isomorphic to  $\mathbb{Q}$ . If  $R$  is the ring of Gaussian integers, then  $F$  is a copy of  $a + bi$  where now  $a$  and  $b$  are elements of  $\mathbb{Q}$ .

If  $R = K[x]$ , where  $K$  is a field, then the field of fractions is denote  $K(x)$ . It consists of all rational functions, that is all quotients

$$\frac{f(x)}{g(x)},$$

where  $f$  and  $g$  are polynomials with coefficients in  $R$ .