1. Suppose that $m$ and $n$ are in $M$. Then 
\[
\phi(m + n) = r(m + n) \\
= rm + rn \\
= \phi(m) + \phi(n).
\]
Thus $\phi$ is additive. Now suppose that $s \in R$. Then 
\[
\phi(sm) = r(sm) \\
= (rs)m \\
= s(rm) \\
= s\phi(m).
\]
Thus $\phi$ is $R$-linear.

2. Let $N$ be a submodule of $M$. Then $N$ is an additive subgroup of $M$ and so it is non-empty and closed under addition. It is also closed under multiplication by definition of the inherited rule for multiplication. Now suppose that $N$ is non-empty and closed under addition and scalar multiplication. As $N$ is non-empty and closed under addition, it follows that it is an additive subgroup. The other axioms obviously hold in $N$, since they hold in the larger set $M$. Thus $N$ is a submodule.

3. Let $K$ be the kernel of $\phi$. As $\phi$ is a homomorphism of the underlying additive groups, it follows that $K$ is an additive subgroup. Suppose that $r \in R$ and $k \in K$. We have 
\[
\phi(rk) = r\phi(k) \\
= r \cdot 0 \\
= 0.
\]
Thus $rk \in K$. It follows that $K$ is closed under scalar multiplication. Therefore $K$ is a submodule.

4. Let $M_i$ be a collection of submodules of an $R$-module $M$ and let $N$ be their intersection. Then $N$ is an additive subgroup as each $M_i$ is an additive subgroup. Suppose that $r \in R$ and $n \in N$. Then for every $i \in I$, $n \in M_i$. As $M_i$ is an $R$-module, it follows that $rn \in M_i$. As this is true for every $i$, in fact $rn \in N$. Thus $N$ is closed under scalar multiplication and so it is a submodule.
5. Let $M_i, i \in I$ be the set of all submodules of $M$ that contain $X$. Then $N$, the intersection of every $M_i$ is a submodule of $M$, which contains $X$. As $N \subseteq M_i$ it is clearly the smallest such submodule.

6. Let $F$ be the set of all functions from $X$ to $M$. We need to define a rule of addition and scalar multiplication. Suppose that $f$ and $g$ are elements of $M$. Define $f + g$ as the pointwise sum, so that

$$(f + g)(x) = f(x) + g(x).$$

Similarly, given $r \in R$ and $f \in F$, define $rf$ as the pointwise product,

$$(rf)(x) = r(f(x)).$$

It is an easy matter to check that with this rule of addition and scalar multiplication, $F$ becomes an $R$-module.

7. Let $H = \text{Hom}_R(M, N)$ be the set of all $R$-module homomorphisms. Then $H$ is a subset of $F$, the set of all functions from $M$ to $N$. It suffices to prove that $H$ is non-empty and closed under addition and scalar multiplication.

First note that the zero map, which sends every element of $M$ to the zero element of $N$, is $R$-linear. Thus $H$ is certainly non-empty. Suppose that $f$ and $g$ are elements of $H$. We need to prove that $f + g$ is $R$-linear. Let $m$ and $n$ be elements of $M$ and $r$ and $s$ be elements of $R$. We have

$$(f + g)(rm + sn) = f(rm + sn) + g(rm + sn)$$

$$= rf(m) + sf(n) + rg(m) + sg(n)$$

$$= rf(m) + rg(m) + sf(m) + sf(n)$$

$$= r(f + g)(m) + s(f + g)(n).$$

Thus $f + g$ is indeed $R$-linear. It is equally easy and just as formal to prove that $rf$ is $R$-linear. Thus $H$ is closed under addition and scalar multiplication and so $H$ is an $R$-module.

8. Since the arbitrary intersection of ideals is an ideal, it suffices to prove that $I$ is an ideal, in the case that $X$ contains one point $x$. Clearly $0 \in I$. Thus $I$ is non-empty. Suppose that $i$ and $j$ are elements of $I$. Then

$$(i + j)x = ix + jx$$

$$= 0 + 0 = 0.$$
Thus $i + j \in I$ and $I$ is closed under addition. Now suppose that $r \in R$ and $i \in I$. Then

$$ri(x) = r(ix) = r0 = 0.$$ 

Thus $ri \in I$ and $I$ is an ideal. Here is another way to conclude that $I$ is an ideal. Let

$$\phi: R \rightarrow \text{Hom}_R(M, M)$$

be the natural map which sends an element $R$ to the $R$-linear map, $m \rightarrow rm$. It is easy to see that $\phi$ is $R$-linear. Replacing $M$ by the module generated by $X$, note that an element $r \in R$ is in $I$ if and only if $\phi(r)$ is the zero map. Thus $I$ is the kernel of $\phi$. It also follows that $I$ is also the annihilator of $\langle X \rangle$. 