13. Homomorphisms and kernels

It is somewhat suprising that one can relax the condition that ϕ is a bijection in the definition of an isomorphism and come up with a very interesting property:

Definition 13.1. *A map $\phi\colon G \longrightarrow H$ between two groups is a* ***homorphism*** *if for every g and h in G,*

$$\phi(gh) = \phi(g)\phi(h).$$

Here is an interesting example of a homomorphism. Define a map

$$\phi\colon G \longrightarrow H$$

where $G = \mathbb{Z}$ and H is a subgroup of order two, say $\mathbb{Z}/2\mathbb{Z}$, by the rule

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd.} \end{cases}$$

We check that ϕ is a homomorphism. Suppose that x and y are two integers. There are four cases. x and y are even, x is even, y is odd, x is odd, y is even, and x and y are both odd.

Now if x and y are both even or both odd, then $x + y$ is even. In this case $\phi(x+y) = 0$. In the first case $\phi(x)+\phi(y) = 0+0 = 0$ and in the second case $\phi(x) + \phi(y) = 1 + 1 = 0$.

Otherwise one is even and the other is odd and $x + y$ is odd. In this case $\phi(x + y) = 1$ and $\phi(x) + \phi(y) = 1 + 0 = 1$. Thus we get a homomorphism.

Here are some elementary properties of homomorphisms.

Lemma 13.2. *Let $\phi\colon G \longrightarrow H$ be a homomorphism.*

(1) $\phi(e) = f$, that is, ϕ maps the identity in G to the identity in H.
(2) $\phi(a^{-1}) = \phi(a)^{-1}$, that is, ϕ maps inverses to inverses.
(3) If K is subgroup of G then $\phi(K)$ is a subgroup of H.

Proof. Let $a = \phi(e)$, where e is the identity in G. Then

$$\begin{aligned} a &= \phi(e) \\ &= \phi(ee) \\ &= \phi(e)\phi(e) \\ &= aa. \end{aligned}$$

Thus $a^2 = a$. Cancelling we get $a = f$, the identity in H. Hence (1).

Let $b = a^{-1}$.

$$\begin{aligned}\phi(e) &= \phi(ab) \\ &= \phi(a)\phi(b).\end{aligned}$$

But then $\phi(b)$ is the inverse of $\phi(a)$, so that $\phi(a^{-1}) = \phi(a)^{-1}$. Hence (2).

Let $X = \phi(K)$. It suffices to check that X is non-empty and closed under products and inverses. X contains f the identity of H, by (1). X is closed under inverses by (2) and closed under products, almost by definition. Thus X is a subgroup. □

Instead of looking at the image, it turns out to be much more interesting to look at the inverse image of the identity.

Definition-Lemma 13.3. *Let $\phi\colon G \longrightarrow H$ be a group homomorphism. The **kernel of** ϕ, denoted $\operatorname{Ker}\phi$, is the inverse image of the identity.*

Then $\operatorname{Ker}\phi$ is a subgroup of G.

Proof. We have to show that the kernel is non-empty and closed under products and inverses.

Note that $\phi(e) = f$ by (13.2). Thus $\operatorname{Ker}\phi$ is certainly non-empty. Now suppose that a and b are in the kernel, so that $\phi(a) = \phi(b) = f$.

$$\begin{aligned}\phi(ab) &= \phi(a)\phi(b) \\ &= ff = f.\end{aligned}$$

So the kernel is closed under products.

Finally suppose that $\phi(a) = f$. Then $\phi(a^{-1}) = \phi(a)^{-1} = f$, where we used (13.2). Thus the kernel is closed under inverses, and so the kernel is a subgroup. □

Here are some basic results about the kernel.

Lemma 13.4. *Let $\phi\colon G \longrightarrow H$ be a homomrphism.*

Then f is injective if and only if $\operatorname{Ker}\phi = \{e\}$.

Proof. If f is injective, then at most one element can be sent to the identity $f \in H$. Since $\phi(e) = f$, it follows that $\operatorname{Ker}\phi = \{e\}$.

Now suppose that $\operatorname{Ker}\phi = \{e\}$ and suppose that $\phi(x) = \phi(y)$. Let $g = x^{-1}y$. Then $\phi(g) = \phi(x^{-1}y) = \phi(x)^{-1}\phi(y) = f$. Thus g is in the kernel of ϕ and so $g = e$. But then $x^{-1}y = e$ and so $x = y$. □

It turns out that the kernel of a homomorphism enjoys a much more important property than just being a subgroup.

Definition 13.5. *Let G be a group and let H be a subgroup of G.*

*We say that H is **normal** in G and write $H \lhd G$, if for every $g \in G$, $gHg^{-1} \subset H$.*

In other words H is normal in G if and only if it is a union of conjugacy classes.

Lemma 13.6. *Let $\phi\colon G \longrightarrow H$ be a homomorphism.*

Then the kernel of ϕ is a normal subgroup of G.

Proof. We have already seen that the kernel is a subgroup. Suppose that $g \in G$. We want to prove that

$$g(\operatorname{Ker} \phi)g^{-1} \subset \operatorname{Ker} \phi.$$

Suppose that $h \in \operatorname{Ker} \phi$. We need to prove that $ghg^{-1} \in \operatorname{Ker} \phi$.

Now

$$\begin{aligned} \phi(ghg^{-1}) &= \phi(g)\phi(h)\phi(g)^{-1} \\ &= \phi(g)f\phi(g)^{-1} \\ &= \phi(g)\phi(g)^{-1} = f. \end{aligned}$$

Thus $ghg^{-1} \in \operatorname{Ker} \phi$. □

It is interesting to look at some examples of subgroups, to see which are normal and which are not.

Lemma 13.7. *Let G be an abelian group and let H be any subgroup.*

Then H is normal in G.

Proof. Clear, as for every $h \in H$ and $g \in G$,

$$ghg^{-1} = gg^{-1}h = h \in H.$$ □

So let us look at the first interesting example of a group which is not abelian.

Take $G = D_3$. Let us first look at $H = \{I, R, R^2\}$. Then H is normal in G. In fact, pick $g \in D_3$. If g belongs to H, there is nothing to prove. Otherwise g is a flip. Let us suppose that it is F_1. Now pick $h \in H$ and consider ghg^{-1}. If $h = I$ then it is clear that $ghg^{-1} = I \in H$.

So suppose that $h = R$. Then

$$\begin{aligned} ghg^{-1} &= F_1RF_1 \\ &= R^2 \in H. \end{aligned}$$

Similarly, if $h = R^2$, then $ghg^{-1} = R \in H$.

Thus H is normal in G.

Now suppose that $H = \{I, F_1\}$. Take $h = F_1$ and $g = R$. Then

$$\begin{aligned} ghg^{-1} &= RF_1R^2 \\ &= F_2. \end{aligned}$$

So $ghg^{-1} \neq H$.

In fact, all of this is much easier to see with S_3. In the first case we are looking at $H = \{e, (1,2,3), (1,3,2)\}$. In this case H is in fact a union of conjugacy classes. (Recall that the conjugacy classes of S_n are entirely determined by the cycle type). So H is obviously normal. Now take $H = \{e, (1,2)\}$, and let $g = (2,3)$. Then

$$\begin{aligned} gHg^{-1} &= \{geg^{-1}, g(1,2)g^{-1}\} \\ &= \{e, (1,3)\}. \end{aligned}$$

Thus H is not normal in this case.

Lemma 13.8. *Let H be a subgroup of a group G.*

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1. *H is normal in G.*
2. *For every $g \in G$, $gHg^{-1} = H$.*
3. *$Ha = aH$, for every $a \in G$.*
4. *The set of left cosets is equal to the set of right cosets.*
5. *H is a union of conjugacy classes.*

Proof. Suppose that (1) holds. Suppose that $g \in G$. Then

$$gHg^{-1} \subset H.$$

Now replace g with k, then

$$kHk^{-1} \subset H,$$

for all $k \in G$. Now take $k = g^{-1}$. Then

$$g^{-1}Hg \subset H,$$

so that

$$H \subset gHg^{-1}.$$

But then (2) holds.

If (2) holds, then (3) holds, simply by multiplying the equality

$$aHa^{-1} = H,$$

on the right by a.

If (3) holds, then (4) certainly holds.

Suppose that (4) holds. Let $g \in G$. Then $g \in gH$ and $g \in Hg$. If the set of left cosets is equal to the set of right cosets, then this means

$gH = Hg$. Now take this equality and multiply it on the right by g^{-1}. Then certainly $gHg^{-1} \subset H$, so that H is normal in G. Hence (1).

Thus (1), (2), (3) and (4) are all equivalent.

Suppose that (5) holds. Then $H = \cup A_i$, where A_i are conjugacy classes. Then

$$\begin{aligned} gHg^{-1} &= \bigcup gA_i g^{-1} \\ &= \bigcup A_i \\ &= H. \end{aligned}$$

Thus H is normal.

Finally suppose that (2) holds. Suppose that $a \in H$ and that A is the conjugacy class to which a belongs. Pick $b \in A$. Then there is an element $g \in G$ such that $gag^{-1} = b$. Then $b \in gHg^{-1} = H$. So $A \subset H$. But then H is a union of conjugacy classes. □

Given this, we can give one more interesting example of a normal subgroup.

Let $G = S_4$. Then let $H = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. We have already seen that H is a subgroup of G. On the other hand, H is a union of conjugacy classes. Indeed the three non-trivial elements of H represent the only permutations with cycle type $(2,2)$. Thus H is normal in G.