15. A little Category Theory

Definition 15.1. *A category $\mathcal{C}$ consists of two things. The first is the* ***objects*** *of the category. The second is, given any two objects, X and Y, we associate a collection of* ***morphisms****, denoted* $\operatorname{Hom}(X, Y)$. *If the collection of all morphisms from X to Y is a set, we say that the category is* ***locally small****; it is* ***small*** *if in addition the objects are a set. Given three objects, X, Y and Z and two morphisms $f \in Hom(X, Y)$, $g \in \operatorname{Hom}(Y, Z)$, there is a rule of composition,*

$$g \circ f \in \operatorname{Hom}(X, Z).$$

A category satisfies the following two axioms.

(1) *Composition is associative. That is, given $f \in \operatorname{Hom}(X, Y)$, $g \in \operatorname{Hom}(Y, Z)$, $h \in \operatorname{Hom}(Z, W)$, we have*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(2) *There is an element $I_X \in \operatorname{Hom}(X, X)$ that acts as an identity, so that if $f \in \operatorname{Hom}(X, Y)$ and $g \in \operatorname{Hom}(Y, X)$ we have,*

$$f \circ I_X = f,$$

and

$$I_X \circ g = g.$$

Given $f \in \operatorname{Hom}(X, Y)$ we say that $g \in \operatorname{Hom}(Y, X)$ is the ***inverse*** *of f if $f \circ g = I_Y$ and $g \circ f = I_X$. In this case we say that f is an* ***isomorphism*** *and we say that X and Y are* ***isomorphic****.*

Given most advanced mathematics classes, there is normally a naturally associated category.

For us, the most natural category is the category of groups. The objects are the collection of all groups and the morphisms are homomorphisms. Composition of morphisms is composition of functions. It is not hard to see that all the axioms are satisfied. In fact the only thing we have not checked is that composition of homomorphisms is a homomorphism, which is an exercise for the reader.

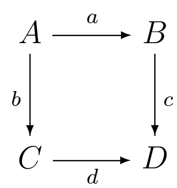
Two groups are isomorphic if and only if they are isomorphic as objects of the category.

Another very natural category is the category of sets. Here a morphism is any function. Two sets are isomorphic if and only if they have the same cardinality.

Yet another category is the category of topological spaces. The objects are topological spaces and the morphisms are continuous maps. Two topological spaces are then homeomorphic if and only if they are isomorphic as objects of this category. The category of metric spaces is a subcategory of the category of topological spaces.

The category of vector spaces, is the category whose objects are vector spaces and whose morphisms are linear maps.

Definition 15.2. *Let*



be a collection of objects and morphisms belonging to a category.

We say that the diagram ***commutes*** *if the two morphisms from* A *to* D *are the same, that is*

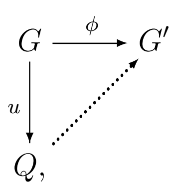
$$c \circ a = d \circ b.$$

In a category, the focus of interest is not the objects, but the morphisms between the objects. In this sense, we would like to characterise the notion of the quotient group in a way that does not make explicit reference to the elements of G/H, but rather define everything in terms of homomorphisms between groups. Even though this is somewhat abstract, there is an obvious advantage to this approach; as a set G/H is rather complicated, its elements are left cosets, which are themselves sets. But really we only need to know what G/H is up to isomorphism.

Definition 15.3. *Let* G *be a group and let* H *be a normal subgroup.*

The ***categorical quotient*** *of* G *by* H *is a group* Q *together with a homomorphism* $u\colon G \longrightarrow Q$*, such that kernel of* u *contains* H*, which is universal amongst all such homomorphisms in the following sense:*

Suppose that $\phi\colon G \longrightarrow G'$ *is any homomorphism such that the kernel of* ϕ *contains* H*. Then there is a unique induced homomorphism*



which makes the diagram commute.

Note that in the definition of the categorical quotient, the most important part of the definition refers to the homomorphism u, and the universal property that it satisfies.

Theorem 15.4. *The category of groups admits categorical quotients.*

That is to say, given a group G and a normal subgroup H, there is a categorical quotient group Q. Furthermore, Q is unique, up to a unique isomorphism.

Proof. We first prove existence. Let G/H be the quotient group and let $u\colon G \longrightarrow G/H$ be the natural homomorphism. I claim that this pair forms a categorical quotient. Thus we have to prove that u is universal.

To this end, suppose that we are given a homomorphism $\phi\colon G \longrightarrow G'$. Define a map

$$f\colon G/H \longrightarrow G'$$

by sending gH to $\phi(g)$. It is clear that the condition that the diagram commutes, forces this definition of f. We have to check that f is well-defined.

Suppose that $g_1H = g_2H$. We need to check that $\phi(g_1) = \phi(g_2)$. As $g_1H = g_2H$, it follows that $g_2 = g_1h$, for some $h \in H$. In this case

$$\begin{aligned}\phi(g_2) &= \phi(g_1h)\\ &= \phi(g_1)\phi(h)\\ &= \phi(g_1),\end{aligned}$$

where we use the fact that h is in the kernel of ϕ. Thus the map f is well-defined and f is unique.

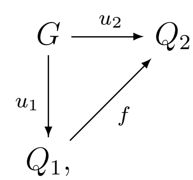
Now we check that f is a homomorphism. Suppose that x and y are in G/H. Then $x = g_1H$ and $y = g_2H$, for some g_1 and g_2 in G. In this case

$$\begin{aligned}f(xy) &= f(g_1g_2H)\\ &= \phi(g_1g_2)\\ &= \phi(g_1)\phi(g_2)\\ &= f(g_1H)f(g_2H) = f(x)f(y).\end{aligned}$$

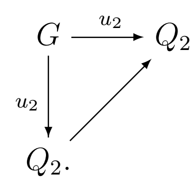
Thus f is a homomorphism. Finally we check that the diagram above commutes. Suppose that $g \in G$. Going along the top we get $\phi(g)$. Going down first, we get gH and then going diagonally we get $f(gH) = \phi(g)$, by definition of f.

Thus G/H is a categorical quotient. In particular categorical quotients exist.

Now we prove that categorical quotients are unique, up to unique isomorphism. Suppose that Q_1 and Q_2 are two such categorical quotients. As Q_1 is a categorical quotient and there is a homomorphism $u_2\colon G \longrightarrow Q_2$ whose kernel contains H, it follows that there is an induced homomorphism $f\colon Q_1 \longrightarrow Q_2$, which makes the following diagram commute



By symmetry there is a homomorphism $g\colon Q_2 \longrightarrow Q_1$, which makes the same diagram commute, with 1 and 2 switched. Consider the composition $f \circ g\colon Q_2 \longrightarrow Q_2$. This is a homomorphism which make the following diagram commute



However there is one homomorphism that makes the diagram commute, namely the identity. By uniqueness, $f \circ g$ must be the identity. Similarly $g \circ f$ must be the identity. So f and g are inverses of each other, and hence isomorphisms. Note that f itself is unique, since its existence was given to us by the universal property of u_1.

Thus the quotient is unique, up to a unique isomorphism. □

Now it is easy to us this to deduce the isomorphism Theorems.

We have already seen that given any group G and a normal subgroup H, there is a natural homomorphism $\phi\colon G \longrightarrow G/H$, whose kernel is H. In fact we will see that this map is not only natural, it is in some sense the only such map.

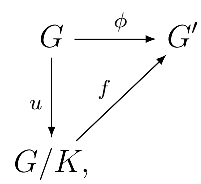
Theorem 15.5 (First Isomorphism Theorem)**.** *Let $\phi\colon G \longrightarrow G'$ be a homomorphism of groups. Suppose that ϕ is onto and let H be the kernel of ϕ.*

Then G' is isomorphic to G/H.

Proof. By the universal property of a quotient, there is a natural homorphism

$$f\colon G/H \longrightarrow G'.$$

As f makes the following diagram commute,



it follows that f is surjective. It remains to prove that f is injective. Suppose that x is in the kernel of f. Then x has the form gH and by definition of f, $f(x) = \phi(g)$. Thus g is in the kernel of ϕ and so $g \in H$. In this case $x = H$, the identity of G/H. So the kernel of f is trivial and f is injective. Hence f is an isomorphism. $\square$

Definition 15.6. *Let G be a group and let H and K be two subgroups of G.*

$H \vee K$ denotes the subgroup generated by the union of H and K.

In general, it is hard to identify $H \vee K$ as a set. However,

Theorem 15.7 (Second Isomorphism Theorem)**.** *Let G be a group, let H be a subgroup and let N be a normal subgroup. Then*

$$H \vee N = HN = \{\, hn \mid h \in H, n \in N \,\}.$$

Furthermore $H \cap N$ is a normal subgroup of H and the two groups $H/H \cap N$ and HN/N are isomorphic.

Proof. The pairwise products of the elements of H and N are certainly elements of $H \vee N$. Thus the RHS of the equality above is a subset of the LHS. The RHS is clearly non-empty, it contains H and N and so it suffices to prove that the RHS is closed under products and inverses.

Suppose that x and y are elements of the RHS. Then $x = h_1 n_1$ and $y = h_2 n_2$, where $h_i \in H$ and $n_i \in N$. Now $h_2^{-1} n_1 h_2 = n_3 \in N$, as N is normal in G. So $n_1 h_2 = h_2 n_3$. In this case

$$\begin{aligned} xy &= (h_1 n_1)(h_2 n_2) \\ &= h_1 (n_1 h_2) n_2 \\ &= h_1 (h_2 n_3) n_2 \\ &= (h_1 h_2)(n_3 n_2), \end{aligned}$$

which shows that xy has the correct form. On the other hand, suppose $x = hn$. Then $hnh^{-1} = m \in N$ as N is normal and so $hn^{-1}h^{-1} = m^{-1}$. In this case

$$\begin{aligned} x^{-1} &= n^{-1} h^{-1} \\ &= h m^{-1}, \end{aligned}$$

so that x^{-1} is of the correct form.

Hence the first statement. Let $H \longrightarrow HN$ be the natural inclusion. As N is normal in G, it is certainly normal in HN, so that we may compose the inclusion with the natural projection map to get a homomorphism

$$H \longrightarrow HN/N$$

This map sends h to hN.

Suppose that $x \in HN/N$. Then $x = hnN = hN$, where $h \in H$. Thus the homorphism above is clearly surjective. Suppose that $h \in H$ belongs to the kernel. Then $hN = N$, the identity coset, so that $h \in N$. Thus $h \in H \cap N$. The result then follows by the First Isomorphism Theorem applied to the map above. □

It is easy to prove the Third isomorphism Theorem from the First.

Theorem 15.8 (Third Isomorphism Theorem)**.** *Let $K \subset H$ be two normal subgroups of a group G.*

Then

$$G/H \simeq (G/K)/(H/K).$$

Proof. Consider the natural map $G \longrightarrow G/H$. The kernel, H, contains K. Thus, by the universal property of G/K, it follows that there is a homomorphism $G/K \longrightarrow G/H$.

This map is clearly surjective. In fact, it sends the left coset gK to the left coset gH. Now suppose that gK is in the kernel. Then the left coset gH is the identity coset, that is $gH = H$, so that $g \in H$. Thus the kernel consists of those left cosets of the form gK, for $g \in H$, that is H/K. The result now follows by the first Isomorphism Theorem. □