## 16. Characteristic subgroups and Products

Recall that a subgroup is normal if it is invariant under conjugation. Now conjugation is just a special case of an automorphism of G.

**Definition 16.1.** Let G be a group and let H be a subgroup. We say that H is a **characteristic subgroup** of G, if for every automorphism  $\phi$  of G,  $\phi(H) = H$ .

It turns out that most of the *general* normal subgroups that we have defined so far are all in fact characteristic subgroups.

**Lemma 16.2.** Let G be a group and let Z = Z(G) be the centre. Then Z is characteristically normal.

*Proof.* Let  $\phi$  be an automorphism of G. We have to show  $\phi(Z) = Z$ . Pick  $z \in Z$ . Then z commutes with every element of G. Pick an element x of G. As  $\phi$  is a bijection,  $x = \phi(y)$ , for some  $y \in G$ .

We have

$$x\phi(z) = \phi(y)\phi(z)$$
$$= \phi(yz)$$
$$= \phi(zy)$$
$$= \phi(z)\phi(y)$$
$$= \phi(z)x.$$

As x is arbitrary, it follows that  $\phi(z)$  commutes with every element of G. But then  $\phi(z) \in Z$ . Thus  $\phi(Z) \subset Z$ . Applying the same result to the inverse  $\psi$  of  $\phi$  we get  $\phi^{-1}(Z) = \psi(Z) \subset Z$ . But then  $Z \subset \phi(Z)$ , so that  $Z = \phi(Z)$ .

**Definition 16.3.** Let G be a group and let x and y be two elements of G.  $x^{-1}y^{-1}xy$  is called the commutator of x and y.

The commutator subgroup of G is the group generated by all of the commutators.

**Lemma 16.4.** Let G be a group and let H be the commutator subgroup.

Then H is characteristically normal in G and the quotient group G/H is abelian. Moreover this quotient is universal amongst all abelian quotients in the following sense.

Suppose that  $\phi: G \longrightarrow G'$  is any homomorphism of groups, where G' is abelian. Then there is a unique homomorphism  $G/H \longrightarrow G'$ .

*Proof.* Suppose that  $\phi$  is an automorphism of G and let x and y be two elements of G. Then

$$\phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y).$$

The last expression is clearly the commutator of  $\phi(x)$  and  $\phi(y)$ . Thus  $\phi(H) \subset H$ . Let  $\psi$  be the inverse of  $\phi$ . Then  $\psi(H) \subset H$  so that  $H \subset \phi(H)$ . Thus  $\phi(H) = H$ . Thus H is characteristically normal in G.

Suppose that aH and bH are two left cosets. Then

$$\begin{aligned} (bH)(aH) &= baH \\ &= ba(a^{-1}b^{-1}ab)H \\ &= abH = (aH)(bH). \end{aligned}$$

Thus G/H is abelian. Suppose that  $\phi: G \longrightarrow G'$  is a homomorphism, and that G' is abelian. By the universal property of a quotient, it suffices to prove that the kernel of  $\phi$  must contain H.

Since H is generated by the commutators, it suffices to prove that any commutator must lie in the kernel of  $\phi$ . Suppose that x and y are in G.

Then  $\phi(x)\phi(y) = \phi(y)\phi(x)$ . It follows that

$$\phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = \phi(x^{-1}y^{-1}xy)$$

is the identity in G'. Thus  $x^{-1}y^{-1}xy$  is sent to the identity, that is, the commutator of x and y lies in the kernel of  $\phi$ .

## **Definition-Lemma 16.5.** Let G and H be any two groups.

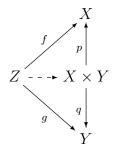
The **product** of G and H, denoted  $G \times H$ , is the group, whose elements are the ordinary elements of the cartesian product of G and H as sets, with multiplication defined as

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2).$$

*Proof.* We need to check that with this law of multiplication,  $G \times H$  becomes a group. This is left as an exercise for the reader.

**Definition 16.6.** Let C be a category and let X and Y be two objects of C. The **categorical product** of X and Y, denoted  $X \times Y$ , is an object together with two morphisms  $p: X \times Y \longrightarrow X$  and  $q: X \times Y \longrightarrow Y$  that are universal amongst all such morphisms, in the following sense.

Suppose that there are morphisms  $f: Z \longrightarrow X$  and  $g: Z \longrightarrow Y$ . Then there is a unique morphism  $Z \longrightarrow X \times Y$  which makes the following diagram commute,



Note that, by the universal property of a categorical product, in any category, the product is unique, up to unique isomorphism. The proof proceeds exactly as in the proof of the uniqueness of a categorical quotient and is left as an exercise for the reader.

## **Lemma 16.7.** The product of groups is a categorical product.

That is, given two groups G and H, the group  $G \times H$  defined in (16.5) satisfies the universal property of (16.6).

*Proof.* First of all note that the two ordinary projection maps  $p: G \times H \longrightarrow G$  and  $q: G \times H \longrightarrow H$  are both homomorphisms (easy exercise left for the reader).

Suppose that we are given a group K and two homomorphisms  $f: K \longrightarrow G$  and  $g: K \longrightarrow H$ . We define a map  $u: K \longrightarrow G \times H$  by sending k to (f(k), g(k)).

It is left as an exercise for the reader to prove that this map is a homomorphism and that it is the only such map, for which the diagram commutes.  $\hfill \Box$