16. Characteristic subgroups and Products

Recall that a subgroup is normal if it is invariant under conjugation. Now conjugation is just a special case of an automorphism of G.

Definition 16.1. *Let G be a group and let H be a subgroup. We say that H is a* ***characteristic subgroup*** *of G, if for every automorphism ϕ of G, $\phi(H) = H$.*

It turns out that most of the *general* normal subgroups that we have defined so far are all in fact characteristic subgroups.

Lemma 16.2. *Let G be a group and let $Z = Z(G)$ be the centre.*

Then Z is characteristically normal.

Proof. Let ϕ be an automorphism of G. We have to show $\phi(Z) = Z$. Pick $z \in Z$. Then z commutes with every element of G. Pick an element x of G. As ϕ is a bijection, $x = \phi(y)$, for some $y \in G$.

We have

$$\begin{aligned} x\phi(z) &= \phi(y)\phi(z) \\ &= \phi(yz) \\ &= \phi(zy) \\ &= \phi(z)\phi(y) \\ &= \phi(z)x. \end{aligned}$$

As x is arbitrary, it follows that $\phi(z)$ commutes with every element of G. But then $\phi(z) \in Z$. Thus $\phi(Z) \subset Z$. Applying the same result to the inverse ψ of ϕ we get $\phi^{-1}(Z) = \psi(Z) \subset Z$. But then $Z \subset \phi(Z)$, so that $Z = \phi(Z)$. □

Definition 16.3. *Let G be a group and let x and y be two elements of G. $x^{-1}y^{-1}xy$ is called the commutator of x and y.*

The ***commutator subgroup*** *of G is the group generated by all of the commutators.*

Lemma 16.4. *Let G be a group and let H be the commutator subgroup.*

Then H is characteristically normal in G and the quotient group G/H is abelian. Moreover this quotient is universal amongst all abelian quotients in the following sense.

Suppose that $\phi\colon G \longrightarrow G'$ is any homomorphism of groups, where G' is abelian. Then there is a unique homomorphism $G/H \longrightarrow G'$.

Proof. Suppose that ϕ is an automorphism of G and let x and y be two elements of G. Then

$$\phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y).$$

The last expression is clearly the commutator of $\phi(x)$ and $\phi(y)$. Thus $\phi(H) \subset H$. Let ψ be the inverse of ϕ. Then $\psi(H) \subset H$ so that $H \subset \phi(H)$. Thus $\phi(H) = H$. Thus H is characteristically normal in G.

Suppose that aH and bH are two left cosets. Then

$$\begin{aligned}(bH)(aH) &= baH \\ &= ba(a^{-1}b^{-1}ab)H \\ &= abH = (aH)(bH).\end{aligned}$$

Thus G/H is abelian. Suppose that $\phi\colon G \longrightarrow G'$ is a homomorphism, and that G' is abelian. By the universal property of a quotient, it suffices to prove that the kernel of ϕ must contain H.

Since H is generated by the commutators, it suffices to prove that any commutator must lie in the kernel of ϕ. Suppose that x and y are in G.

Then $\phi(x)\phi(y) = \phi(y)\phi(x)$. It follows that

$$\phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = \phi(x^{-1}y^{-1}xy)$$

is the identity in G'. Thus $x^{-1}y^{-1}xy$ is sent to the identity, that is, the commutator of x and y lies in the kernel of ϕ. □

Definition-Lemma 16.5. *Let G and H be any two groups.*

The ***product*** *of G and H, denoted $G \times H$, is the group, whose elements are the ordinary elements of the cartesian product of G and H as sets, with multiplication defined as*

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2).$$

Proof. We need to check that with this law of multiplication, $G \times H$ becomes a group. This is left as an exercise for the reader. □

Definition 16.6. *Let $\mathcal{C}$ be a category and let X and Y be two objects of $\mathcal{C}$. The* ***categorical product*** *of X and Y, denoted $X \times Y$, is an object together with two morphisms $p\colon X \times Y \longrightarrow X$ and $q\colon X \times Y \longrightarrow Y$ that are universal amongst all such morphisms, in the following sense.*

Suppose that there are morphisms $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$. Then there is a unique morphism $Z \longrightarrow X \times Y$ which makes the following diagram commute,

$$\begin{tikzcd}
& X \\
Z \arrow[ur, "f"] \arrow[r, dashed] \arrow[dr, "g"'] & X \times Y \arrow[u, "p"'] \arrow[d, "q"] \\
& Y
\end{tikzcd}$$

Note that, by the universal property of a categorical product, in any category, the product is unique, up to unique isomorphism. The proof proceeds exactly as in the proof of the uniqueness of a categorical quotient and is left as an exercise for the reader.

Lemma 16.7. *The product of groups is a categorical product.*

That is, given two groups G and H, the group $G \times H$ defined in (16.5) *satisfies the universal property of* (16.6).

Proof. First of all note that the two ordinary projection maps $p\colon G \times H \longrightarrow G$ and $q\colon G \times H \longrightarrow H$ are both homomorphisms (easy exercise left for the reader).

Suppose that we are given a group K and two homomorphisms $f\colon K \longrightarrow G$ and $g\colon K \longrightarrow H$. We define a map $u\colon K \longrightarrow G \times H$ by sending k to $(f(k), g(k))$.

It is left as an exercise for the reader to prove that this map is a homomorphism and that it is the only such map, for which the diagram commutes. $\square$