# 17. The Alternating Groups

Consider the group $S_3$. Then this group contains a normal subgroup, generated by a 3-cycle.

Now the elements of $S_3$ come in three types. The identity, the product of zero transpositions; the transpositions, the product of one transposition, and the three cycles, products of two transpositions. Then the normal subgroup above, consists of all permutations that can be represented as a product of an even number of transpositions.

In general there is no canonical way to represent a permutation as a product of transpositions. But we might hope that the pattern above continues to hold in every permutation group.

**Definition 17.1.** *Let $\sigma \in S_n$ be a permutation.*

*We say that $\sigma$ is* ***even*** *if it can be represented as a product of an even number of transpositions. We say that $\sigma$ is* ***odd*** *if it can be represented as a product of an odd number of transpositions.*

The following result is much trickier to prove than it looks.

**Lemma 17.2.** *Let $\sigma \in S_n$ be a permutation.*

*Then $\sigma$ is not both an even and an odd permutation.*

There is no entirely satisfactory proof of (17.2). Here is perhaps the simplest.

**Definition 17.3.** *Let $x_1, x_2, \dots, x_n$ be indeterminates and set*

$$f(x_1, x_2, \dots, x_n) = \prod_{i<j} (x_i - x_j).$$

For example, if $n = 3$, then

$$f(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

**Definition 17.4.** *Given a permutation $\sigma \in S_n$, let*

$$g = \sigma^*(f) = \prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)}).$$

Suppose that $\sigma = (1, 2) \in S_3$. Then

$$g = \sigma^*(f) = (x_2 - x_1)(x_2 - x_3)(x_1 - x_3) = -(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = -f.$$

The following Lemma is the key part of the proof of (17.2).

**Lemma 17.5.** *Let $\sigma$ and $\tau$ be two permutations and let $\rho = \sigma\tau$. Then*

*(1) $\sigma^*(f) = \pm f$.*
*(2) $\rho^*(f) = \sigma^*(\tau^*(f))$.*
*(3) $\sigma^*(f) = -f$, whenever $\sigma$ is a transposition.*

*Proof.* $g$ is clearly a product of terms of the form $x_i - x_j$ or $x_j - x_i$. Thus $g = \pm f$. Hence (1).

$$\begin{aligned}\sigma^*(\tau^*(f)) &= \sigma^*(\prod_{i<j}(x_{\tau(i)} - x_{\tau(j)}))\\ &= \prod_{i<j}(x_{(\sigma(\tau(i))} - x_{\sigma(\tau(j))})\\ &= \prod_{i<j}(x_{\rho(i)} - x_{\rho(j)})\\ &= \rho^*(f).\end{aligned}$$

Hence (2).

Suppose that $\sigma = (i, j)$, where $i < j$. Then the only terms of $f$ affected by $\sigma$ are the ones that involve either $x_i$ or $x_j$. Suppose $a \neq i$, $a \neq j$. If $a < i$, then $x_a - x_i$ is sent to $x_a - x_j$ and there is no change of sign. If $i < a < j$ then $x_i - x_a$ is sent to $x_j - x_a = -(x_a - x_j)$. Thus there is a change in sign. If $j < a$ then $x_i - x_a$ is sent $x_j - a$ and there is no change in sign. Similarly if we consider $x_j - x_a$ then we get a similiar result and the changes in sign will cancel out. Otherwise we need to consider $x_i - x_j$ when there is one change of sign. In total then the sign changes. Hence (3). □

*Proof.* Suppose that $\sigma$ is a product of an even number of transpositions. Then by (2) and (3) of (17.5), $\sigma^*(f) = f$. Similarly if $\sigma^*(f)$ is a product of an odd number of transpositions, then $\sigma^*(f) = -f$. Thus $\sigma$ cannot be both even and odd. □

**Definition-Lemma 17.6.** *There is a surjective homomorphism*

$$\phi\colon S_n \longrightarrow \mathbb{Z}_2$$

*The kernel consists of the even transpositions, and is called the* ***alternating group*** $A_n$.

*Proof.* The map sends an even transposition to 0 and an odd transposition to 1. (2) of (17.5) implies that this map is a homomorphism. □

Note that half of the elements of $S_n$ are even, so that the alternating group $A_n$ contains $\frac{n!}{2}$. One of the most important properties of the alternating group is,

**Theorem 17.7.** *Suppose that* $n \geq 5$.

*The only normal subgroup of* $S_n$ *is* $A_n$. *Moreover* $A_n$ *is simple, that is,* $A_n$ *has no proper normal subgroups.*

Recall that $V = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is a normal subgroup of $S_4$. It is also therefore a normal subgroup of $A_4$.