MODEL ANSWERS TO THE FOURTH HOMEWORK

2. Chapter 3, Section 1: 1 (a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 2 & 1 & 3 & 6 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}.$$

5. It suffices to find the cycle type and take the lowest common multiples of the individual lengths of a cycle decomposition.
(a)

$$(1, 4)(2, 5, 3)$$

Order 6.
(b)

$$(1, 3, 2)$$

Order 3.
(c)

$$(2, 4)$$

Order 2.
2. Chapter 3, Section 2: 1 As σ and τ are cycles, we may find integers $a_1, a_2, \dots, a_k$ and $b_1, b_2, \dots, b_l$ such that $\sigma = (a_1, a_2, \dots, a_k)$ and $\tau = (b_1, b_2, \dots, b_l)$. To say that σ and τ are disjoint cycles is equivalent to saying that the two sets $S = \{a_1, a_2, \dots, a_k\}$ and $T = \{b_1, b_2, \dots, b_l\}$ are disjoint.
We want to prove that

$$\sigma\tau = \tau\sigma.$$

As both sides of this equation are permutations of the first n natural numbers, it suffices to show that they have the same effect on any integer $1 \leq j \leq n$.
If j is not in $S \cup T$, then there is nothing to prove; both sides clearly fix j. Otherwise $j \in S \cup T$. By symmetry we may asume $j \in S$. As S and T are disjoint, it follows that $j \notin T$.

As $j \in S$, $j = a_i$, some i. Then $\sigma(a_i) = a_{i+1}$, where we take $i+1$ modulo k (that is we adopt the convention that $k+1=1$). In this case $a_{i+1} \in S$ so $a_{i+1} \notin T$ as well. Thus both sides send $j = a_i$ to a_{i+1}. Thus both sides have the same effect on j, regardless of j and so

$$\sigma\tau = \tau\sigma.$$

2. Chapter 3, Section 2: 2
(a)

$$(1,3,4,2)(5,7,9)$$

Order 12.
(b)

$$(1,7)(2,6)(3,5).$$

Order 2.
(c)

$$(1,6)(2,5)(3,7)$$

Order 2.
2. Chapter 3, Section 2:
3 (a)

$$(2,4,1)(3,5,7,6).$$

Order 12.
(f)

$$(1,4,2,5,3)$$

Order 5.
2. Chapter 3, Section 2: 8 (a)

$$(2,1)(2,4)(3,6)(3,7)(3,5).$$

(f)

$$(1,3)(1,5)(1,2)(1,4).$$

3. Easy, the conjugate is $(2,7,5,3)(1,6,4)$. The order of σ is 12 and the order of τ is three.
4. There are quite a few possibilities for τ. One obvious one is

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 5 & 4 & 7 & 6 \end{pmatrix}.$$

5. Let $H = \langle (1,2)(1,2,3,\dots,n) \rangle$. We want to show that H is the whole of S_n. As the transpositions generate S_n, it suffices to prove that every transposition is in H.
Now the idea is that it is very hard to compute products in S_n, but it is easy to compute conjugates. So instead of using the fact that H is closed under products and inverses, let us use the fact that it is closed

under taking conjugates (clear, as a conjugate is a product of elements of H and their inverses).
Since conjugation preserves cycle type, we start with the transposition $\sigma = (1,2)$ (in fact this is the only place to start).
To warm up, consider conjugating σ with $\tau = (1,2,3,\dots,n)$. The conjugate is $(2,3)$. Thus H must contain $(2,3)$.
Given that H contains $(2,3)$ it must contain the conjugate of $(2,3)$ by τ, which is $(3,4)$ (or what comes to the same thing, H must contain the conjugate of $(1,2)$ by τ^2).
Continuing in this way, it is clear that H (by an easy induction in fact) must contain every transposition of the form $(i, i+1)$ and of course the last one, $(n,1) = (1,n)$.
From here, let us try to show that H contains every transposition of the form $(1,i)$. For example, to get $(1,3)$, start with $(1,2)$ and conjugate it by $(2,3)$. Suppose, by way of induction, that H contains $(1,i)$. Then H must contain the conjugate of $(1,i)$ by $(i,i+1)$ which is $(1,i+1)$. Thus by induction H contains every transposition of the form $(1,i)$.
Now we are almost home. Note that H must contain every transposition of the form $(2,j)$. Indeed $(2,j)$ is the conjugate of $(1,j)$ by the transposition $(1,2)$.
Now consider an aribtrary transposition (i,j). This is the conjugate of $(1,2)$ by the element $(1,i)(2,j)$. Thus H contains every transposition.
Aliter:
There is another way to show that the transpositions $(i,i+1)$, $1 \leq i \leq n$ generate S_n. Consider a deck of cards in the order given by a permutation $\tau \in S_n$. It is enough to show that we can put the deck of cards into the correct order, just using $(i,i+1)$, $1 \leq i \leq n$.
Suppose that we have rearranged the cards so that the first k cards are in the correct order. By induction it is enough to show we can put the $(k+1)$th card into the $(k+1)$th position.
Consider the $(k+1)$th card. Suppose it occupies position l. If $l = k+1$ we are done. Now $l > k$ since the first k cards are in their correct position. Thus $l > k+1$. If we apply the transposition $(l-1,l)$ then we put the $(k+1)$th card into the $(l-1)$th position. Continuing in this way, we can continue swapping until it is in the $(k+1)$th position.
It follows that we can undo any permutation by applying a sequence of transpositions $\tau_1, \tau_2, \dots, \tau_k$ of the form $(i,i+1)$,

$$\tau^{-1} = \tau_1 \tau_2 \dots \tau_k.$$

Taking inverses we express τ as product in the opposite order.