MODEL ANSWERS TO THE FIFTH HOMEWORK

1. Chapter 3, Section 5: 1 (a) Yes. Given a and $b \in \mathbb{Z}$,

$$\begin{aligned}\phi(ab) &= [ab] \\ &= [a][b] \\ &= \phi(a)\phi(b).\end{aligned}$$

This map is clearly surjective but not injective. Indeed the kernel is easily seen to be $n\mathbb{Z}$.

(b) No. Suppose that G is not abelian and that $xy \neq yx$. Then $x^{-1}y^{-1} \neq y^{-1}x^{-1}$. On the other hand

$$\begin{aligned}\phi(xy) &= (xy)^{-1} \\ &= y^{-1}x^{-1} \\ &\neq x^{-1}y^{-1} \\ &= \phi(x)\phi(y),\end{aligned}$$

and one wrong certainly does not make a right.

(c) Yes. Suppose that x and y are in G. As G is abelian

$$\begin{aligned}\phi(xy) &= (xy)^{-1} \\ &= y^{-1}x^{-1} \\ &= x^{-1}y^{-1} \\ &= \phi(x)\phi(y).\end{aligned}$$

Thus ϕ is a homomorphism. Suppose that $a \in G$. Then a is the inverse of $b = a^{-1}$, so that $\phi(b) = a$. Thus ϕ is surjective. Suppose that a is in the kernel of ϕ. Then $a^{-1} = e$ and so $a = e$. Thus the kernel of ϕ is trivial and ϕ is injective.

(d) Yes. ϕ is a homomorphism as the product of two positive numbers is positive, the product of two negative numbers is positive and the product of a negative and a positive number is negative.

This map is clearly surjective. The kernel consists of all positive real numbers. Thus ϕ is far from injective.

(e) Yes. Suppose that x and y are in G. Then

$$\begin{aligned}\phi(xy) &= (xy)^n \\ &= x^n y^n \\ &= \phi(x)\phi(y).\end{aligned}$$

In general this map is neither injective nor surjective. For example, if $G = \mathbb{Z}$ and $n = 2$ then the image of ϕ is $2\mathbb{Z}$, and for example 1 is not in the image.
Now suppose that $G = \mathbb{Z}_4$ and $n = 2$. Then $2[2] = [4] = [0]$, so that $[2]$ is in the kernel.
10. We need to check that $aHa^{-1} = H$ for all $a \in G$. If we pick $a \in H$ there is nothing to prove. Now $a = f^i g^j$. Conjugation by a is the same as conjugation by g^j followed by conjugation by f^i. So we only need to worry about conjugation by f. Now $gf = fg^{-1}$ so that $fgf^{-1} = g^{-1}$. Thus conjugation by f leaves H fixed, as it sends a generator to a generator.
12. Let $g \in G$. We want to show that $gZg^{-1} \subset Z$. Pick $z \in Z$. Then z commutes with g, so that $gzg^{-1} = zgg^{-1} = z \in Z$. Thus Z is normal in G.
17. Let $g \in G$. We want to show that $g(M \cap N)g^{-1} \subset M \cap N$. Pick $h \in M \cap N$. Then $h \in M$ and $h \in N$. It follows that $ghg^{-1} \in M$ and $ghg^{-1} \in N$, as both M and N are normal in G. But then $ghg^{-1} \in M \cap N$ and so $M \cap N$ is normal.
22. $H = \{i, (1,2)\}$. Then the left cosets of H are
(a)

$$\begin{aligned} H &= \{e, (1,2)\} \\ (1,3)H &= \{(1,3), (1,2,3)\} \\ (2,3)H &= \{(2,3), (1,3,2)\} \end{aligned}$$

and the right cosets are
(b)

$$\begin{aligned} H &= \{e, (1,2)\} \\ H(1,3) &= \{(1,3), (1,3,2)\} \\ H(2,3) &= \{(2,3), (1,2,3)\}. \end{aligned}$$

(c) Clearly not every left coset is a right coset. For example $\{(1,3), (1,2,3)\}$ is a left coset, but not a right coset.
27. Let $g \in G$. We have to show that $g\theta(N)g^{-1} \subset \theta(N)$. Now as θ is surjective, we may write $g = \theta(h)$, for some $h \in G$. Pick $m \in \theta(N)$. Then $m = \theta(n)$, for some $n \in N$. We have

$$\begin{aligned} gmg^{-1} &= \theta(h)\theta(n)\theta(h)^{-1} \\ &= \theta(hnh^{-1}). \end{aligned}$$

Now $hnh^{-1} \in N$ as N is normal. So $gmg^{-1} \in \theta(N)$ and $\theta(N)$ is normal in G.
37. Note that S_3 is the group of permutations of three objects. So we want to find three things on which G acts. Pick any element h of G. Then the order of h divides the order of G. As the order of G is six, it follows that the order of h is one, two, three, or six. It cannot be six, as then G would be cyclic, whence abelian, and it can only be one if h is the identity.
Note that elements of order 3 come in pairs. If a is an element of order 3 then $a^2 = a^{-1}$ also has order three and they are the two elements of $\langle a \rangle$ not equal to the identity. So the number of elements of order 3 is even. As there are five elements of G which don't have order one, it follows that at least one element a of G has order 2. If $H = \langle a \rangle$ then H is a subgroup of G of order two.
Let b be any other element of G. Consider the subgroup $K = \langle a, b \rangle$ of G generated by a and b. Then K has at least three elements, e, a and b and on the other hand the order of K is even by Lagrange as H is a subgroup of order 2. Thus K has at least four elements. As the order of K divides the order of G the order of K is six, so that $G = \langle a, b \rangle$ is generated by a and b.
If $ab = ba$ it is not hard to check that G is abelian. As G is not abelian we must have $ab \neq ba$.
As H is a subgroup of G of order two, the number of left cosets of H in G (the index of H in G) is equal to three, by Lagrange. Let S be the set of left cosets. Define a map from G to $A(S)$,

$$\phi \colon G \longrightarrow A(S)$$

by sending g to $\sigma = \phi(g)$, where σ is the map,

$$\sigma \colon S \longrightarrow S$$

$\sigma(xH) = gxH$, that is, σ acts on the left cosets by left multiplication by g. If $xH = yH$ so that $y = xh$ for some $h \in H$ then

$$gy = g(xh) = (gx)h,$$

so that $(gy)H = (gx)H$ and σ is well-defined. σ is a bijection, as its inverse τ is given by left multiplication by g^{-1}. Now we check that ϕ is a homomorphism. Suppose that g_1 and g_2 are two elements of G. Set $\sigma_i = \phi(g_i)$ and let $\tau = \phi(g_1 g_2)$. We need to check that $\tau = \sigma_1 \sigma_2$. Pick a left coset xH. Then

$$\begin{aligned}\sigma_1\sigma_2(xH) &= \sigma_1(g_2xH)\\ &= g_1g_2xH\\ &= \tau(xH).\end{aligned}$$

Thus ϕ is a homomorphism.
We check that ϕ is injective. It suffices to prove that the kernel of ϕ is trivial. Pick $g \in \operatorname{Ker}\phi$. Then $\sigma = \phi(g)$ is the identity permutation, so that for every left coset xH,

$$gxH = xH.$$

Consider the left coset H. Then $gH = H$. It follows that $g \in H$, so that either $g = e$ or $g = a$. If $g = a$, then consider the left coset bH. We would then have $abH = bH$, so that $ab = bh'$, where $h' \in H$. So $h' = e$ or $h' = a$. If $h' = e$, then $ab = b$, and $a = e$, a contradiction. Otherwise $ab = ba$, a contradiction. Thus $g = e$, the kernel of ϕ is trivial and ϕ is injective. As $A(S)$ has order six and ϕ is injective, it follows that ϕ is a bijection.
Thus G is isomorphic to S_3.
43. Let G be a group of order nine. Let $g \in G$ be an element of G. Then the order of g divides the order of G. Thus the order of g is 1, 3 or 9. If G is cyclic then G is certainly abelian. Thus we may assume that there is no element of order nine. On the other hand the order of g is one if and only if $g = e$.
Thus we may assume that every element of G, other than the identity, has order three. Let $a \in G$ be an element of G, other than the identity. Let $H = \langle a \rangle$. Then H has order three. Let S be the set of left cosets of H in G. By Lagrange S has three elements. Let

$$\phi\colon G \longrightarrow A(S) \simeq S_3$$

be the corresponding homomorphism. Let G' be the order of the image. Then the order of G' is the number of left cosets of the kernel, which divides G by Lagrange. On the other hand the order of G' divides the order of $A(S)$, again by Lagrange.
Thus G' must have order three. It follows that the kernel of ϕ has order three. Thus the kernel of ϕ is H and H is a normal subgroup of G.
Let $b \in G$ be any element of G that does not commute with a. Then bab^{-1} must be an element of H, as H is normal in G, and so $bab^{-1} = a^2$.

It follows that $ba = a^2b^2$. In this case

$$\begin{aligned}(ba)^2 &= baba\\ &= a^2b^2ba\\ &= a^2a\\ &= e.\end{aligned}$$

Thus ba is an element of order 2, which is impossible as G has order 9.
49. Let S be the set of left cosets of H in G. Define a map

$$\phi\colon G \longrightarrow A(S)$$

by sending $g \in G$ to the permutation $\sigma \in A(S)$, a map

$$\sigma\colon S \longrightarrow S$$

defined by the rule $\sigma(aH) = gaH$. Note that τ, which acts by multiplication on the left by g^{-1} is the inverse of σ, so that σ is indeed a permutation of S. It is easy to check, as before, that ϕ is a homomorphism.
Let N be the kernel of ϕ. Then N is normal in G. Suppose that $a \in N$ and let $\sigma = \phi(a)$. Then σ is the identity permutation of S. In particular $\sigma(H) = H$, so that $aH = H$. Thus $a \in H$ and so $N \subset H$.
Let n be the index of H, so that the image of G has at most $n!$ elements. In this case there are at most $n!$ left cosets of N in G, since each left coset of N in G is mapped to a different element of $A(S)$. Thus the index of N is at most $n!$.
52. Let A be the set of elements such that $\phi(a) = a^{-1}$. Pick an element $g \in G$ and let $B = g^{-1}A$. Then

$$\begin{aligned}|A \cap B| &= |A| + |B| - |A \cup B|\\ &> (3/4)|G| + (3/4)|G| - |G|\\ &= (1/2)|G|.\end{aligned}$$

Now pick $h \in A \cap B$ and suppose that $g \in A$. Then $gh \in A$. It follows that

$$\begin{aligned}h^{-1}g^{-1} &= (gh)^{-1}\\ &= \phi(gh)\\ &= \phi(g)\phi(h)\\ &= g^{-1}h^{-1}.\end{aligned}$$

Taking inverses, we see that g and h must commute. Let C be the centraliser of g. Then $A \cap B \subset C$, so that C contains more than half the elements of G. On the other hand, C is subgroup of G. By

Lagrange the order of C divides the order of G. Thus $C = G$. Hence g is in the centre Z of G and so the centre Z of G contains at least $3/4$ of the elements of G. But then the centre of G must also equal G, as it is also a subgroup of G. Thus G is abelian.