MODEL ANSWERS TO THE SIXTH HOMEWORK

1. Chapter 3, Section 6: 1 There are two cosets. The first coset is $[1] = N$, the second is the coset containing -1, which is the set of all negative real numbers.
$[1] \cdot [1] = [1]$, $[1] \cdot [-1] = [-1] \cdot [1] = [-1]$ and $[-1] \cdot [-1] = [1]$.
Chapter 3, Section 6: 2. Let $a \in \mathbb{R}$. Then $[a] = \{a, -a\}$. Thus any coset contains two elements, exactly one of which is a positive real number. Given a and b positive, $[a][b] = [ab]$. Define a homomorphism

$$\phi\colon G \longrightarrow \mathbb{R}^+,$$

by sending a to $|a|$. The kernel is $N = \{1, -1\}$. By the first Isomorphism Theorem, $G/N \simeq \mathbb{R}^+$.
Chapter 3, Section 6: 3. Consider the canonical homomorphism

$$u\colon G \longrightarrow G/N.$$

Then $M = u^{-1}(\bar{M})$. As the kernel of u is N, it follows that M contains N, as $\bar{M}$ contains the identity of G/N.
To show that M is a subgroup of G, it suffices to prove that it is closed under products and inverses. Suppose that a and b are in M. Then $u(a)$ and $u(b)$ are in $\bar{M}$. Then $u(ab) = u(a)u(b) \in \bar{M}$ as $\bar{M}$ is closed under products.
Thus $ab \in M$ and M is closed under products.
Similarly $u(a^{-1}) = u(a)^{-1} \in \bar{M}$ as $\bar{M}$ is closed under inverses. Thus $a^{-1} \in M$ and M is closed under inverses.
Thus M is a subgroup of G.
Chapter 3, Section 6: 4. Suppose that $\bar{M}$ is normal in G/N.
Pick $g \in G$. We want to prove $gMg^{-1} \subset M$. Pick $a \in M$. Then

$$\phi(gag^{-1}) = \phi(g)\phi(a)\phi(g)^{-1}.$$

As $\bar{M}$ is normal in G/N, it follows that $\phi(g)\phi(a)\phi(g)^{-1} \in \bar{M}$. But then $gag^{-1} \in M$.
Chapter 3, Section 6: 7. As G is cyclic, G is generated by a single element a. But then G/N is generated by $u(a) = aN$.
Chapter 3, Section 6: 8. Pick two elements of G/N. As G/N is the set of left cosets in G, these two elements have the form aN and bN. It

follows that

$$\begin{aligned}(aN)(bN) &= abN \\ &= baN \\ &= (bN)(aN),\end{aligned}$$

where we use the fact that G is abelian to deduce $ab = ba$.
But then G/N is abelian.
Chapter 3, Section 6: 11. Suppose that G/Z is cyclic. Then there is an element a of G such that aZ generates G/Z, so that every left coset has the form $a^i Z$, for some i. Pick two elements x and y of G. Then $xZ = a^i Z$ and $yZ = a^j Z$, for some i and j, so that $x = a^i z_1$ and $y = a^j z_2$, for $z_i \in Z$, $i = 1, 2$.
Then

$$\begin{aligned}xy &= (a^i z_1)(a^j z_2) \\ &= a^i a^j z_1 z_2 \\ &= a^{i+j} z_1 z_2.\end{aligned}$$

Similarly $yx = a^{i+j} z_1 z_2$. Thus $xy = yx$ and G is abelian.
Chapter 3, Section 6: 12. Suppose that G/N is abelian. Pick a and $b \in G$. Then

$$\begin{aligned}aNbN &= abN \\ &= baN \\ &= bNaN,\end{aligned}$$

so that $abN = baN$ and so $ba = abn$ for some $n \in N$. It follows that $b^{-1}a^{-1}ba = n \in N$. Thus $c^{-1}d^{-1}cd \in N$ for all c and d. Replacing c by a^{-1} and d by b^{-1} we get $aba^{-1}b^{-1} \in N$.
Chapter 3, Section 7: 2. We want to use the First Isomorphism Theorem. Define a homomorphism

$$\phi\colon G \longrightarrow \mathbb{R}$$

by sending f to $\phi(f) = f(1/4)$. Suppose that f and $g \in G$. Then

$$\begin{aligned}\phi(f+g) &= (f+g)(1/4) \\ &= f(1/4) + g(1/4) \\ &= \phi(f) + \phi(g).\end{aligned}$$

Thus ϕ is a homorphism. ϕ is clearly surjective. For example, given a real number a, let f be the constant function $f(x) = a$. Then $\phi(f) = f(1/4) = a$.

The kernel of ϕ consists of all functions that vanish at 1/4, that is N. Thus by the First Isomorphism Theorem, $G/N \simeq \mathbb{R}$.
Chapter 3, Section 7: 4. We first prove (a) and (c). Define a homomorphism

$$\phi\colon G \longrightarrow G_2,$$

by sending $g = (g_1, g_2)$ to g_2. Suppose that $g = (g_1, g_2)$ and $h = (h_1, h_2)$ are in G. Then

$$\begin{aligned}
\phi(gh) &= \phi(g_1h_1, g_2h_2) \\
&= g_2h_2 \\
&= \phi(g_1, g_2)\phi(h_1, h_2) \\
&= \phi(g)\phi(h).
\end{aligned}$$

Thus ϕ is a homomorphism. ϕ is clearly surjective as given $g_2 \in G$, $\phi(e_1, g_2) = g_2$.
Suppose that $(g_1, g_2) \in \operatorname{Ker}\phi$. Then $g_2 = e_2$. Thus $N = \operatorname{Ker}\phi$. Hence (a). (c) follows from the First Isomorphism Theorem. To prove (b), define a homomorphism

$$f\colon N \longrightarrow G_1$$

by sending (g_1, e_2) to g_1. This is clearly an isomorphism.
Chapter 3, Section 7: 6. By definition the order of a is the order of the subgroup $H = \langle a \rangle$ and the order of aN is the order of the subgroup $H' = \langle aN \rangle$. Now it is clear that H' is the image of H under the canonical homomorphism

$$u\colon G \longrightarrow G/N.$$

So it suffices to prove that if we have a surjective homomorphism

$$\phi\colon H \longrightarrow H'$$

then the order of H' divides the order of H. But by the first isomorphism Theorem,

$$H' \simeq H/H'',$$

where H'' is the kernel of ϕ. Thus the order of H' is the index of H'' in H, the number of left cosets of H'' in H, which by Lagrange divides the order of H.