

MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. For Chapter 3, Section 3: 1. (a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix} = (1, 2, 4)(3, 5)(6, 7, 8, 9),$$

a product of $2 + 1 + 3 = 6$ transpositions. Even.

(b) $5 + 2 = 7$, odd.

(c) $5 + 5 = 10$, even.

(d) $1 + 2 + 1 + 2 + 2 = 8$, even.

1. For Chapter 3, Section 3: 2. We have already seen that a k -cycle is a product of $k - 1$ transpositions. Thus the result is clear.

1. For Chapter 3, Section 3: 3. Clear, since they have the same cycle type.

1. For Chapter 3, Section 3: 5. First the bottom row is missing 4 and 5. Let us try

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 4 & 5 & 7 & 8 & 9 & 6 \end{pmatrix} = (1, 3, 2)(6, 7, 8, 9).$$

This is a product of $5 = 2 + 3$ transpositions. Thus this permutation is odd. Now consider

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 5 & 4 & 7 & 8 & 9 & 6 \end{pmatrix}.$$

This differs from the first permutation by a transposition. Hence it is even, as the other is odd.

1. For Chapter 3, Section 3: 6. Since every element of A_n is a product of an even number of transpositions, pairing off the transpositions arbitrarily, it suffices to prove that the product of two transpositions is a product of 3-cycles.

Suppose that (a, b) and (c, d) are two transpositions. There are three cases. If $\{a, b\} = \{c, d\}$, then their product is the identity, which is the product of any 3-cycle with its inverse.

Now suppose that $\{a, b\} \cap \{c, d\}$ contains one element, say a . Then

$$(a, b)(a, c) = (a, c, b)$$

a 3-cycle.

Finally suppose that $\{a, b\} \cap \{c, d\}$ is empty. Then

$$(a, b)(c, d) = (a, b)(a, c)(a, c)(c, d) = (a, c, b)(c, d, a),$$

a product of two 3-cycles.

1. For Chapter 3, Section 3: 7. As every element of A_n is a product of 3-cycles, it suffices to prove that every 3-cycle is a product of n -cycles. By symmetry (in fact conjugation by elements of S_n) it suffices to prove that there is a single 3-cycle, which is a product of n -cycles. Consider the elements $\sigma = (1, 2, 3, 4, \dots, n)$ and $\tau = (1, 3, 2, n, n-1, \dots, 4)$. In this case

$$\sigma\tau = (1, 4, 2),$$

as is easily seen by direct computation.

1. For Chapter 3, Section 3: 8. Done in class. Let

$$V = \{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}.$$

Then V is easily seen to be closed under products and inverses. As V is a union of conjugacy classes, it follows that V is normal (not only in A_4 , but even in S_4 in fact).

2. **Bonus Challenge Problems.** For Chapter 3, Section 3: 9. This is a little tricky and there are many different ways to prove this result. Let N be a normal subgroup of A_n and suppose that $N \neq \{e\}$.

Suppose that $\alpha \neq e$ is an element of N , that moves the fewest number of elements. We try to prove that α is in fact a 3-cycle. Suppose not; we will derive a contradiction.

First suppose that in the cycle decomposition of α , we have a cycle of length greater than two. Putting this first and relabelling, we have

$$\alpha = (1, 2, 3, \dots) \dots$$

We may also assume that α does not fix 4. Consider conjugating α by $\beta = (1, 2)(3, 4)$ to get α_1 . Then

$$\alpha_1 = (2, 1, 4, \dots).$$

Note that

$$\gamma = \alpha_1\alpha = \beta\alpha\beta^{-1}\alpha \in N,$$

as N is normal. γ fixes everything that α fixes and also fixes 1, a contradiction, as α is supposed to fix the most elements.

Therefore α is a product of disjoint transpositions:

$$\alpha = (1, 2)(3, 4) \dots$$

Consider conjugating α by $\beta = (3, 4, 5)$ to get α_1 . Then

$$\alpha_1 = (1, 2)(4, 5) \dots,$$

Then

$$\gamma = \alpha_1\alpha = (3, 4)(4, 5) \dots \in N,$$

which fixes 1 and 2 and fixes everything α fixes except possibly 5, a contradiction.

Thus α is a 3-cycle.

For Chapter 3, Section 3: 10. This follows *almost* immediately from 9 and 6. In S_n , the conjugacy classes are determined by cycle type. In particular any two 3-cycles are conjugate in S_n . We want to show that they are conjugate in A_n . Now suppose that σ and σ' are two 3-cycles and that $\tau \in S_n$ conjugates σ to σ' . If τ is even there is nothing to prove. Otherwise pick a transposition τ_1 that is disjoint from the numbers that σ permutes (possible as $n \geq 5$). Then $\tau\tau_1$ is even and is easily seen to still conjugate σ to σ' .

Let N be a normal subgroup of A_n . Suppose that $N \neq \{e\}$. Then by the previous question N must contain a 3-cycle. As N is normal, it must contain every 3-cycle, as any two 3-cycles are conjugate. But then by 6, $N = A_n$.