PRACTICE PROBLEMS FOR THE SECOND MIDTERM

1. Give the definition of:

(i) the centraliser of an element of a group.
(ii) the centre of a group.
(iii) the Euler phi-function.
(iv) a permutation.
(v) the group S_n.
(vi) the cycle type of a permutation.
(vii) conjugation.
(viii) conjugacy classes.
(ix) an isomorphism of groups.
(x) an automorphism.
(xi) a group homomorphism.
(xii) the kernel.
(xiii) a normal subgroup.
(xiv) the quotient group.
(xv) a category.
(xvi) the commutator subgroup.

2. Compute
(i) $5^{103} \mod 3$.
(ii) $14^{204} \mod 15$
(iii) the cycle decomposition of the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 5 & 6 & 4 & 2 & 3 & 8 \end{pmatrix}.$$

(iv) $\tau\sigma\tau^{-1}$, where $\sigma = (1,3,5)(2,4,6,8)$.
3. Let G be a group. Show that every subgroup of index 2 is normal.
4. Let $G = S_3$.
(i) Show that $H = \langle(1,2,3)\rangle$ is a normal subgroup of G.
(ii) Show that $H = \langle(1,2)\rangle$ is not a normal subgroup of G.
5. Let G be a group and let H be a subgroup. Prove that the following are equivalent.

(1) H is normal in G.
(2) For every $g \in G$, $gHg^{-1} = H$.
(3) For every $a \in G$, $aH = Ha$.
(4) The set of left cosets is equal to the set of right cosets.

6. Prove the Second Isomorphism Theorem.

7. Let G be the group of nonzero real numbers under multiplication and let $N = \{ -1, 1\}$. Prove that G/N is isomorphic to the positive real numbers under multiplication.