You have three hours.

There are 10 problems, and the total number of points is 145. Show all your work. *Please make your work as clear and easy to follow as possible.*

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>145</td>
<td></td>
</tr>
</tbody>
</table>
1. (30pts) (i) *Give the definition of the absolute value of a real number.*

If \( r \) is a real number and \( r \) is non-negative then the absolute value of \( r \) is \( r \); if \( r \) is negative the absolute value is \(-r\).

(ii) *Give the definition of the difference of two sets.*

If \( A \) and \( B \) are two sets the difference \( A \setminus B \) is the set of elements of \( A \) which are not elements of \( B \).

(iii) *Give the definition of the symmetric difference of two sets.*

If \( A \) and \( B \) are two sets then the symmetric difference is the union of \( A \setminus B \) and \( B \setminus A \).
(iv) *Give the definition of a surjective function.*

A function $f: A \rightarrow B$ is surjective if for every $b \in B$ we may find $a \in A$ such that $f(a) = b$.

(v) *Give the definition of a bijective function.*

$f: A \rightarrow B$ is a bijective function if $f$ is injective and surjective.

(vi) *Give the definition of an invertible function.*

$f: A \rightarrow B$ is an invertible function if there is a function $g: B \rightarrow A$ such that $f \circ g = \text{id}_B: B \rightarrow B$ and $g \circ f = \text{id}_A: A \rightarrow A$. 
2. (10pts) (i) Prove that \(|r|^2 = r^2\) for all real numbers \(r\).

There are two cases. If \(r \geq 0\) then \(|r| = r\) and so

\[ |r|^2 = r^2. \]

If \(r < 0\) then \(|r| = -r\) and so

\[ |r|^2 = (-r)^2 \]
\[ = r^2. \]

Either way, \(|r|^2 = r^2\).

(ii) Prove that if \(n\) is an integer then \(n(n + 1)\) is even.

There are two cases. If \(n\) is even then \(n = 2k\). In this case

\[ n(n + 1) = 2k(n + 1) \]

is even as \(k(n + 1)\) is an integer.

Now suppose that \(n\) is odd. Let

\[ A = \{ a \mid a \text{ is even and } a \leq n \}. \]

Let \(m\) be the largest element of \(A\) and let \(r = n - m\). Note that \(m = 2k\), for some \(k\). \(r \geq 0\) as \(m \leq n\). If \(r = 0\) then \(m = n\) is odd, a contradiction. If \(r \geq 2\) then \(m + 2 = 2k + 2 = 2(k + 1)\) is even and \(m + 2 \leq m + r = n\) so that \(m < m + 2 \in A\). This contradicts the fact that \(m\) is the largest element of \(A\). It follows that \(r = 1\).

Thus \(n = 2k + 1\). In this case

\[ n(n + 1) = n(2k + 2) \]
\[ = 2n(k + 1), \]

is even, as \(n(k + 1)\) is an integer.

Either way, \(n(n + 1)\) is even.
3. (10pts) Let $x$ and $y$ be positive real numbers. Show that

$$\sqrt{xy} \geq \frac{2}{\frac{1}{x} + \frac{1}{y}}.$$ 

We have

$$(x - y)^2 = |x - y|^2 \geq 0.$$ 

Thus

$$x^2 - 2xy + y^2 \geq 0.$$ 

Adding $4xy$ to both sides we get

$$(x + y)^2 = x^2 + 2xy + y^2 \geq 4xy.$$ 

As $x$ and $y$ are positive real numbers, we have $x + y \geq 0$. Thus taking the square root of both sides

$$x + y \geq 2\sqrt{xy}.$$ 

Multiplying both sides by $\sqrt{xy} > 0$ we have

$$(x + y)\sqrt{xy} \geq 2xy.$$ 

Dividing both sides by $x + y > 0$ we have

$$\frac{\sqrt{xy}}{x + y} \geq \frac{2}{\frac{1}{x} + \frac{1}{y}}.$$
4. (10pts) **Prove that**

\[ n^3 \leq 2^n, \]

for all integers \( n \geq 10. \)

Let \( P(n) \) be the statement that

\[ n^3 \leq 2^n. \]

We prove that \( P(n) \) holds for all natural numbers \( n \) at least ten. If \( n = 10 \) then the LHS is

\[ n^3 = 10^3 = 2^3 \cdot 5^3 \]

and the RHS is

\[ 2^{10} = 2^3 \cdot 2^7. \]

Now \( 5^3 = 125 \) and \( 2^7 = 128 \), so that the

\[ 2^3 \cdot 5^3 \leq 2^3 \cdot 2^7. \]

Thus \( P(10) \) holds.

Suppose that \( P(k) \) holds and \( k \geq 9. \) We check that \( P(k+1) \) holds. We have

\[
(k + 1)^3 = k^3 + 3k^2 + 3k + 1 \\
\leq k^3 + 3k^2 + 3k^2 + 3k^2 \\
= k^3 + 9k^2 \\
\leq k^3 + k^3 \\
\leq 2^k + 2^k \\
= 2 \cdot 2^k \\
= 2^{k+1},
\]

where we used the fact that \( P(k) \) holds to get from line four to line five. Thus \( P(k+1) \) holds.

As we have checked that \( P(10) \) holds and \( P(k) \implies P(k+1), \) mathematical induction implies that \( P(n) \) holds for all \( n \geq 10, \) that is

\[ n^3 \leq 2^n \]

for all integers \( n \geq 10. \)
5. (10pts) If 
\[ A \subset \{1, 2, 3, \ldots, n\} \]
then prove that 
\[ |A| \text{ is even if and only if } |A \triangle \{1\}| \text{ is odd.} \]

First observe that \( n \) is even if and only if \( n + 1 \) is odd.
Suppose that \( |A| = m \).
There are two cases. If \( 1 \notin A \) then
\[
A \triangle \{1\} = A \setminus \{1\} \cup \{1\} \setminus A \\
= A \cup \{1\}.
\]
In this case
\[
|A \triangle \{1\}| = |A \cup \{1\}| \\
= |A| + 1 \\
= m + 1.
\]
Thus \(|A|\) is even if and only if \(|A \triangle \{1\}|\) is odd
If \( 1 \in A \) then
\[
A \triangle \{1\} = A \setminus \{1\} \cup \{1\} \setminus A \\
= A \setminus \{1\}.
\]
In this case
\[
|A \triangle \{1\}| = |A \setminus \{1\}| \\
= |A| - 1 \\
= m - 1.
\]
As \( m - 1 \) is even if and only if \( m \) is odd, it follows that \( m - 1 \) is odd if
and only if \( m \) is even. Thus \(|A|\) is even if and only if \(|A \triangle \{1\}|\) is odd
Either way, \(|A|\) is even if and only if \(|A \triangle \{1\}|\) is odd.
6. (10pts) Prove or disprove:

\[ \forall \epsilon > 0, \ \exists N \in \mathbb{Z}, \ (n \geq N) \implies \frac{1000}{n} < \epsilon. \]

This is true and so we prove it. Pick an integer \( N \) such that

\[ N > \frac{1000}{\epsilon}. \]

If \( n > N \) then

\[
\frac{1000}{n} = \frac{1000}{n} \cdot 1 \\
= \frac{1000}{n} \cdot \frac{\epsilon}{\epsilon} \\
= \frac{\epsilon}{n} \cdot \frac{1000}{\epsilon} \\
< \epsilon \cdot \frac{N}{n} \\
< \epsilon \cdot \frac{n}{n} \\
= \epsilon \cdot 1 \\
= \epsilon.
\]
7. (15pts) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions.
   (a) Show that if $f$ and $g$ are injective then $g \circ f$ is injective.

   Suppose that $(g \circ f)(a_1) = g \circ f(a_2)$. Let $b_i = f(a_i)$. We have
   
   
   \[
   g(b_1) = g(f(a_1)) \\
   = (g \circ f)(a_1) \\
   = (g \circ f)(a_2) \\
   = g(f(a_2)) \\
   = g(b_2). 
   \]

   As $g$ is injective, it follows that $b_1 = b_2$. Therefore
   
   \[
   f(a_1) = b_1 \\
   = b_2 \\
   = f(a_2). 
   \]

   As $f$ is injective, it follows that $a_1 = a_2$. Therefore $g \circ f$ is injective.

   (b) Show that if $f$ and $g$ are surjective then $g \circ f$ is surjective.

   Suppose that $c \in C$. As $g$ is surjective, we may find $b \in B$ such that $g(b) = c$. As $f$ is surjective, we may find $a \in A$ such that $f(a) = b$. We have
   
   \[
   (g \circ f)(a) = g(f(a)) \\
   = g(b) \\
   = c. 
   \]

   Thus $g \circ f$ is surjective.

   (c) Show that if $f$ and $g$ are bijective then $g \circ f$ is bijective.

   As $f$ and $g$, $f$ and $g$ are injective and surjective. By (a), $g \circ f$ is injective and by (b) $g \circ f$ is surjective. But then $g \circ f$ is bijective.
8. (20pts) (a) If $A$, $B$ and $C$ are three sets then find a formula for $|A \cup B \cup C|$ and prove your formula is correct.

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap C| - |A \cap B| + |A \cap B \cap C|.$$ We prove this by inclusion-exclusion. Suppose that $x \in A \cup B \cup C$. We consider how many times we count $x$, depending on where it lies. If $x$ belongs to $A$ but not $B$ or $C$ then $x$ gets counted once as an element of $A$ and it is not counted as any element of any other set on the RHS.

If $x$ belongs to $B \cap C$ but not $A$, it gets included once as an element of $B$, once as an element of $C$ and excluded once as an element of $B \cap C$. If $x$ belongs to $A \cap B \cap C$, it gets included three times, as elements of $A$, $B$ and $C$, it gets excluded three times, as elements of $A \cap B$, $B \cap C$ and $A \cap C$ and it is included once as an element of $A \cap B \cap C$.

In all three cases, $x$ is counted once in total. By symmetry $x$ is always counted once. Thus the formula is correct.

(b) How many numbers between 1 and 10,000 are not divisible by at least one of 2, 3 and 5 (you may use the standard properties of primes)?

Let $A$ be the integers between 1 and 10,000 divisible by 2, $B$ be the integers between 1 and 10,000 divisible by 3 and $C$ be the integers between 1 and 10,000 divisible by 5, so that

$$A = \{ k \in \mathbb{Z} \mid 1 \leq k \leq 10,000, \text{ } k \text{ } \text{is divisible by } 2 \}$$

$$B = \{ k \in \mathbb{Z} \mid 1 \leq k \leq 10,000, \text{ } k \text{ } \text{is divisible by } 3 \}$$

$$C = \{ k \in \mathbb{Z} \mid 1 \leq k \leq 10,000, \text{ } k \text{ } \text{is divisible by } 5 \}.$$ We use the formula in (a) to count the number of elements of $A \cup B \cup C$. These are the integers divisible by at least one of 2, 3, or 5. Suppose that $a \in A$. Then we can find $k \in \mathbb{Z}$ such that $a = 2k$. As $1 \leq a \leq 10,000$, we have $1 \leq k \leq 5000$. Thus

$$|A| = 5000$$

Similarly,

$$|C| = 2000.$$
Now $b \in B$ if and only if $b = 3k$, some integer $k$. As $1 \leq b \leq 10,000$, $1 \leq k \leq 3333$. Thus
\[ |B| = 3333. \]

Now
\[ A \cap B = \{ k \in \mathbb{Z} | 1 \leq k \leq 10,000, k \text{ is divisible by } 6 \}. \]
Thus $a \in A \cap B$ if and only if $a = 6k$ for some integer $k$. We have $1 \leq k \leq 1666$. Thus
\[ |A \cap B| = 1666. \]
Similarly
\[ |A \cap C| = 10,000 \quad \text{and} \quad |B \cap C| = 666. \]
Finally,
\[ A \cap B \cap C = \{ k \in \mathbb{Z} | 1 \leq k \leq 10,000, k \text{ is divisible by } 30 \}. \]
Thus $a \in A \cap B \cap C$ if and only if $a = 30k$ for some integer $k$. We have $1 \leq k \leq 333$. Thus
\[ |A \cap B \cap C| = 333. \]
It follows that
\[
|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap C| - |A \cap B| + |A \cap B \cap C| \\
= 5000 + 3333 + 2000 - 1666 - 1000 - 666 + 333 \\
= 7334.
\]
Thus the number of integers between 1 and 10,000 not divisible by one of 2, 3 or 5 is
\[ 7334. \]
9. (20pts) (a) Let \( k \) and \( n \) be natural numbers and suppose that \( k \leq n \). Prove that
\[
\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.
\]
Recall that
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]
We have
\[
\binom{n}{k+1} + \binom{n}{k} = \frac{n!}{(k+1)!(n-k-1)!} + \frac{n!}{k!(n-k)!}
= \frac{n!}{k!(n-k-1)!} \left( \frac{1}{k+1} + \frac{1}{n-k} \right)
= \frac{n!}{k!(n-k-1)!} \left( \frac{n-k}{(k+1)(n-k)} \right)
= \frac{n!}{(n+1)(k+1)(n-k)}
= \frac{(n+1)!}{(k+1)!(n-k)!}
= \binom{n+1}{k+1}.
\]
(b) If $n$ is a natural number and $x$ and $y$ are indeterminates then prove that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{i+j=n} \binom{i+j}{i} x^i y^j.$$  

Let $P(n)$ be the statement that we have equality above. We prove $P(n)$ holds for all natural numbers $n$ by induction on $n$. If $n = 0$ the LHS is $(x + y)^0 = 1$, and the RHS is

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k} = (0)^0 x^0 y^0 = 1.$$  

As we have equality, $P(0)$ holds. Now suppose that $P(m)$ holds. We check that $P(m + 1)$ holds. We have

$$(x + y)^{m+1} = (x + y)(x + y)^m
= (x + y) \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}
= x \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} + y \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}
= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1}
= \sum_{k=1}^{m+1} \binom{m}{k-1} x^k y^{m-k+1} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1}
= y^{m+1} + \sum_{k=1}^{m+1} \left( \binom{m}{k-1} + \binom{m}{k} \right) x^k y^{m-k+1} + x^{m+1}
= \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m-k+1},$$

where we use the inductive hypothesis to get from line one to line two and (a) to get from line seven to line eight. Thus $P(k + 1)$ holds.
Thus \( P(n) \) holds for all natural numbers \( n \), by mathematical induction, that is

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]
10. (10pts) Prove that if \( A \) is a set then \(|A| < 2^{|A|}\).

The function

\[ g: A \rightarrow \mathcal{P}(A), \]

which sends an element \( a \) of \( A \) to the singleton set which contains \( a \),

\[ g(a) = \{ a \} \]

is easily seen to be injective.

Thus if the result does not hold then there is a surjection \( f: A \rightarrow \mathcal{P}(A) \). We will derive a contradiction.

Let

\[ B = \{ a \in A | a \notin f(a) \}. \]

Then \( B \subset A \) so that \( B \in \mathcal{P}(A) \). As \( f \) is surjective, it follows that we may find \( b \in A \) such that \( f(b) = B \).

There are two cases. First suppose that \( b \in B \). Then \( b \in f(b) \) so that \( b \notin B \), by definition of \( B \). This is a contradiction.

Otherwise \( b \notin B \). Then \( b \notin f(b) \) so that \( b \in B \), by definition of \( B \). This is a contradiction.

Either way we get a contradiction and so there is no surjective function. Thus the cardinality of the powerset is greater than the cardinality of \( A \).
Bonus Challenge Problems
11. (10pts) Give a different proof of 9 (a).

We count the number of ways to pick $k + 1$ objects from $n + 1$ objects. Imagine one of the objects is red and the others $n$ objects are blue. If we pick $k + 1$ then either we pick the red object or we don’t. If we do pick it then we have to pick $k$ objects from the remaining $n$ objects. There are

\[
\binom{n}{k}
\]

ways to do this.
If we don’t pick the red object then we have to pick $k + 1$ objects from the remaining $n$ objects. There are

\[
\binom{n}{k + 1}
\]

ways to do this.
Putting all of this together we get the formula:

\[
\binom{n + 1}{k + 1} = \binom{n}{k} + \binom{n}{k + 1}.
\]
12. (10pts) Show that for all non-negative integers \( m \) and \( n \) we have

\[ F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}. \]

where \( F_n \) is the Fibonacci sequence, \( 0, 1, 1, 2, 3, 5, 8, \ldots \).

See the lecture notes.
13. (10pts) Prove that $\mathbb{R}$ is uncountable.

Suppose not, suppose that the real numbers are countable. Then there would be a surjective function $f: \mathbb{N} \longrightarrow (0, 1)$. Then we get a list of all real numbers between 0 and 1, $r_0, r_1, r_2, \ldots$, where $r_i = f(i)$. Imagine making an actual list of these numbers

$$
\begin{align*}
    r_0 &= 0.a_{01}a_{02}a_{03}\ldots \\
    r_1 &= 0.a_{11}a_{12}a_{13}\ldots \\
    \vdots &= \\
    r_n &= 0.a_{n1}a_{n2}a_{n3}\ldots
\end{align*}
$$

We construct another real number $r$,

$$r = 0.a_{12}a_{3}\ldots,$$

as follows.

If the first digit $a_{11}$ of $r_1$ is not one then we let the first digit $a_1$ of $r$ be one. If the first digit $a_{11}$ of $r_1$ is one then we let the first digit $a_1$ of $r_1$ be two.

If the second digit $a_{22}$ of $r_2$ is not one then we let the second digit $a_2$ of $r$ be one. If the first digit $a_{22}$ of $r_2$ is one then we let the first digit $a_2$ of $r_2$ be two.

In general, we define the $n$th digit $a_n$ of $r$ as follows:

$$a_n = \begin{cases} 
1 & \text{if } a_{nn} \neq 1 \\
2 & \text{if } a_{nn} = 1.
\end{cases}$$

As $f$ is surjective there is a natural number $n$ such that $f(n) = r$, that is, $r = r_n$. Suppose that $n = 0$. There are two cases. Either $m = 0$ in which case by definition of $m$, $m_0 \neq 0$. Or $m = 1$ in which case $m_0 = 0$. Either way, $m \neq m_0$, which contradicts the fact that $r = r_0$. Thus $n \neq 1$.

Now suppose that $n > 0$. What is the $n$th digit $a_n$ of $r_n$? If the $n$th digit is 1 then the $n$th digit of $r_n$ is not equal to one. If the $n$th digit is 2 then the $n$th digit of $r_n$ is equal to one. Either way, $a_n \neq a_{nn}$. This contradicts the fact that $r = r_n$. Thus $r$ does not belong to the list of real numbers. This contradicts the fact that $f$ is surjective. Therefore the reals are uncountable.
14. (10pts) Prove that if $f : A \to B$ and $g : B \to A$ are injective
then $|A| = |B|$.

See the lecture notes.