HOMEWORK 3, DUE TUESDAY APRIL 25TH

1. Prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for every positive integer n.
2. (5.1), (5.2), (5.3), (5.4).
3. Define a sequence of numbers $b_1, b_2, \dots$ recursively by the formula:

$$b_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 + \frac{1}{b_{n-1}} & \text{if } n > 1. \end{cases}$$

(i) Write down the first couple of terms of the sequence.
(ii) Prove that for any positive integer n

$$b_n = \frac{F_{n+1}}{F_n},$$

where $F_0, F_1, F_2, \dots$ is the Fibonacci sequence.
(iii) Prove that for any positive integer n

$$b_{n+1} - b_n = \frac{(-1)^{n+1}}{F_n F_{n+1}}.$$

4. Show that we can put the exact postage of at least 34 cents using only stamps of denominations 5 and 9.
Challenge problems/Just for fun:
5. Consider a circle coloured with $2n$ dots, half of which are red and half or which are blue. Starting at some point of the circle and going around the circle clockwise until you get back to where you started, consider the trip a successful one, if at all times the number of red dots you passed is always at least the number of blue dots.
Show that, no matter how you place the $2n$ dots, there is always a place to start a successful trip.
6. Let p be a positive integer. Compute

$$1^p + 2^p + 3^p + 4^p + \cdots + (n-1)^p + n^p,$$

and prove you formula is valid by induction, for as many values of p as you have the patience.
7. Consider the following proof that all cows have the same colour. There are only finitely many cows and so we can prove they have the same colour by induction.

Let $P(n)$ be the proposition that any n cows have the same colour. $P(1)$ is certainly true, since one cow has one colour.
We now prove that $P(k) \implies P(k+1)$. Consider a herd of $k+1$ cows. Pick any cow and consider the remaining k cows. They all have the same colour by induction. Let's say this colour is brown. Now compare the last cow with $k-1$ of these brown cows. Again we have k cows and so by induction they again have the same colour. Since one of these k cows is one of the original brown cows, it follows that these k cows are all brown. In particular the last cow is brown and so all $k+1$ cows are brown. Thus we have shown that $P(k) \implies P(k+1)$.
It follows by induction that all cows have the same colour.
What do you think about this argument?
8. Suppose a prisoner is told that they will be hanged sometime between Monday and Friday of next week. However, the exact day will be a surprise (that is, they will not know the night before that they will be executed the next day). The prisoner, interested in outsmarting their executioner, attempts to determine which day the execution will occur.
They reason that it cannot occur on Friday, since if it had not occurred by the end of Thursday, they would know the execution would be on Friday. Therefore, they can eliminate Friday as a possibility. With Friday eliminated, they decide that it cannot occur on Thursday, since if it had not occurred on Wednesday, they would know that it had to be on Thursday. Therefore, they can eliminate Thursday. This reasoning proceeds until they have eliminated all possibilities. They conclude that they will not be hanged next week.
To their surprise, they are hanged on Wednesday.
What is wrong with the prisoner's argument?