# HOMEWORK 5, DUE TUESDAY MAY 9TH

1. Write down the negation of the following propositions:
(a)

$$\forall \epsilon > 0,\ \exists \delta > 0,\ |x-1| < \delta \implies |x^2-1| < \epsilon.$$

(b)

$$\forall \epsilon > 0,\ \forall x \in \mathbb{R},\ \exists n \in \mathbb{Z},\ |x-n| < \epsilon.$$

(c) Let $\alpha$ be an irrational number, that is, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$,

$$\forall \epsilon > 0,\ \forall x \in \mathbb{R},\ \exists m, n \in \mathbb{Z},\ |x-m-n\alpha| < \epsilon.$$

2. Prove or disprove:
(a)

$$\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ y^2 > 2017 + x.$$

(b)

$$\forall y \in \mathbb{R},\ \exists x \in \mathbb{R},\ y^3 > 2017 + x.$$

(c)

$$\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ y^3 > 2017 + x.$$

(d)

$$\forall \epsilon > 0,\ \exists N \in \mathbb{Z},\ (n \geq N) \implies \frac{1000}{n} < \epsilon.$$

3. Prove that

$$\forall L_1, L_2 \in \mathbb{R} \quad ((\forall \epsilon > 0,\ |L_1 - L_2| < \epsilon) \implies L_1 = L_2).$$

4. Using quantifiers, define what it means for a sequence $x_1, x_2, \dots$ to get closer and closer to $a$.
5. Define a sequence of rational numbers $a_1, a_2, \dots$ recursively by

$$a_n = \begin{cases} 2 & \text{if } n = 1 \\ \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}} & \text{if } n > 1. \end{cases}$$

Prove that
(a) the sequence is bounded from below, by the square root of two.
(b) the sequence is strictly monotonic decreasing, that is, $a_{n+1} < a_n$ for all $n$.
**Challenge problems/Just for fun:**
(c) the infimum is the square root of two.
6. Write down predicates $P(n)$ of the integers, which use only quantifiers, addition and multiplication of integers, such that the set

$$A = \{\, n \in \mathbb{Z} \,|\, P(n) \,\}$$

(i) consists of all powers of 2,

$$A = \{\, \pm 2^k \mid k \in \mathbb{Z} \,\}.$$

(ii) consists of all powers of 5,

$$A = \{\, \pm 5^k \mid k \in \mathbb{Z} \,\}.$$

(iii) consists of all powers of 10,

$$A = \{\, \pm 10^k \mid k \in \mathbb{Z} \,\}$$

(this appears in GEB: parts (i) and (ii) are not so bad, but I have no idea how to solve part (iii)).

7. How many people do you need to invite to a dinner party to ensure that at least three people will be mutual acquaintances, or at least three people will be mutual strangers? As you don't have a big table, you want to find the smallest number of people you can invite while still satisfying these conditions.

8. Find

$$\sqrt{7 + \sqrt{7 + \sqrt{7 + \sqrt{7 + \dots}}}}$$

and

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{\dots}}}}.$$