14. Composition of Functions

One of the most important thing we can do with functions is compose them:

Definition 14.1. *Let* $f\colon A \longrightarrow B$ *and* $g\colon B \longrightarrow C$ *be two functions. The* ***composition*** *of* f *and* g *is a function from* A *to* C*, denoted* $g \circ f\colon A \longrightarrow C$*, defined by the rule:*

$$(g \circ f)(a) = g(f(a)).$$

Note that to compose two functions f and g, in that order, the range of f must equal the domain of g.

Example 14.2. *Suppose that*

$$A = \{ a, b, c \}, \qquad B = \{ 0, 1 \} \qquad \textit{and} \qquad C = \{ \alpha, \beta, \gamma \}.$$

Let

$$f\colon A \longrightarrow B \qquad \text{and} \qquad g\colon B \longrightarrow C$$

be the functions

$$f(a) = 0 \qquad f(b) = 1 \qquad f(c) = 0 \qquad g(0) = \gamma \qquad \text{and} \qquad g(1) = \beta.$$

Then the composition of f and g is the function

$$g \circ f\colon A \longrightarrow C$$

given by

$$(g \circ f)(a) = g(0) = \gamma \qquad (g \circ f)(b) = g(1) = \beta \qquad \text{and} \qquad (g \circ f)(c) = g(0) = \gamma.$$

Note that

$$g \circ f \neq f \circ g$$

in general. In fact, in the example above $f \circ g$ does not even make sense, so not only is the equation not valid,

$$g \circ f = f \circ g$$

the RHS is not even defined.

If $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$ then the composition either way is defined so that both sides of the equation

$$g \circ f = f \circ g$$

makes sense. However the LHS is a function $g \circ f\colon A \longrightarrow A$ and the RHS is a function $f \circ g\colon B \longrightarrow B$, so the two functions are different (at least unless $A = B$).

One reason you can compose either way and get the same domain and range is if $A = B = C$. But even then the composition either way might not be the same.

Example 14.3. *Suppose that*

$$A = \{ a, b \}.$$

Let

$$f\colon A \longrightarrow A \qquad \text{and} \qquad g\colon A \longrightarrow A$$

be the functions

$$f(a) = a \qquad f(b) = a \qquad g(a) = b \qquad \text{and} \qquad g(b) = b.$$

So f is the constant function which sends everything to a and g is the constant function which sends everything to b. Then $g \circ f$ is the constant function which sends everything to b and $f \circ g$ is the constant function which sends everything to a. In particular

$$g \circ f \neq f \circ g,$$

even though both sides make sense, and they have the same domain and range.

The identity function has one very special property with respect to composition:

Lemma 14.4. *If $f\colon A \longrightarrow B$ is any function then*

$$f \circ id_A = f\colon A \longrightarrow B \qquad \textit{and} \qquad id_B \circ f = f\colon A \longrightarrow B$$

Proof. We first show that

$$f \circ \mathrm{id}_A = f$$

Both sides are functions from A to B. Therefore it suffices to show they have the same effect on any element of A. If $a \in A$ then

$$\begin{aligned} (f \circ \mathrm{id}_A)(a) &= f(\mathrm{id}_A(a)) \\ &= f(a). \end{aligned}$$

Thus

$$f \circ \mathrm{id}_A = f$$

Now we show that

$$\mathrm{id}_B \circ f = f.$$

Both sides are functions from A to B. Therefore it suffices to show they have the same effect on any element of A. If $a \in A$ then

$$\begin{aligned} (\mathrm{id}_B \circ f)(a) &= \mathrm{id}_B(f(a))) \\ &= f(a). \end{aligned}$$

□

Definition 14.5. *Let $f\colon A \longrightarrow B$ be a function.*

*We say that f is **invertible** if there is a function $g\colon B \longrightarrow A$ such that $g \circ f = id_A\colon A \longrightarrow A$ and $f \circ g = id_B\colon B \longrightarrow B$. In this case we call g the **inverse** of A.*

Example 14.6. *If*

$$A = \{ a, b \}, \qquad B = \{ 0, 1 \}$$

and $f\colon A \longrightarrow B$ is the function given by $f(a) = 0$, $f(b) = 1$ then f is invertible with inverse $g\colon B \longrightarrow A$ given by $g(0) = a$ and $g(1) = b$.

Note that f sets up a correspondence between the elements of A and the elements of B. In particular, if A and B are finite sets then A and B have the same cardinality.

Definition 14.7. *We say two sets A and B have the same* ***cardinality****, denoted $|A| = |B|$, if there is an invertible function $f\colon A \longrightarrow B$.*